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Faul-Free Hamiltonian Cycle in Faulty Möbius Cubes*

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Abstract

An n-dimensional Möbius cube, MQ_n, is created by rearranging some of the connections of the hypercube, Q_n [Cull95][Fan98]. In this paper, we demonstrate that MQ_n is $(n-3)$-hamiltonian connected and $(n-2)$-hamiltonian. In other words, we prove that there exists a hamiltonian path between any pair of vertices in a faulty MQ_n with $n-3$ faults. We also show that a ring of length $2^n - f_v$ can be embedded in a faulty MQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n-2$ and $n \geq 3$. That is, the faulty MQ_n remains hamiltonian with $n-2$ faults. A recent result has shown that a ring of length $2^n - 2f_v$ can be embedded in a faulty hypercube, if $f_v + f_e \leq n-1$ and $n \geq 4$, with a few additional constraints [Sengupta98]. Our results, in comparison to the hypercube, show that longer rings can be embedded in MQ_n without additional constraints.

Keywords: Möbius cube, fault tolerant, hamiltonian, hamiltonian connected.

1 Introduction

The hypercube is a popular network because of its attractive properties, including regularity, symmetry, powerful computability, strong connectivity, recursive construction, partitionability, and relatively low link complexity [Bhuyan84][Leu99][Sengupta98][Tseng96]. The *Möbius cube* MQ_n is created by rearranging some of the connections of the hypercube Q_n, and the total number of *vertices* and *edges* in a Möbius cube is the same as those of a hypercube. The Möbius cubes have been studied recently because they have several properties that are superior to hypercubes. For example, the diameter of MQ_n is about one half that of Q_n, the average number of communication steps between nodes for MQ_n is about two-thirds of the average for Q_n, and 1-MQ_n has dynamic performance superior to that of Q_n [Cull95][Fan98].

The architecture of an *interconnection network* is usually represented as a *graph*. A *ring* structure (*hamiltonian cycle*) is widely used in interconnection networks, for its good properties such as low connectivity, simplicity, extensibility, and its feasiable implementation. The embedding problem, which maps a source graph into a host graph, is an important and interesting topic of recent studies. Embedding rings into various networks has been discussed. For example, a ring (*fault-tolerant* ring) can be embedded in faulty Stars [Tseng97], faulty arrangement graphs [Hsieh99], double loop networks [Sung98], de Bruijn networks [Rowley93], faulty twisted cubes [Huang99], faulty crossed cubes [Huang99-2], and faulty hypercubes [Leu99][Sengupta98][Tseng96].

A ring of length $2^n - 2f_v$ can be embedded in a faulty hypercube with f_v faulty nodes and f_e faulty edges, if $f_v + f_e \leq n-1$ and $n \geq 4$, with a few additional constraints shown in [Sengupta98]. In this paper, we will prove that there exists a *hamiltonian path* between any

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pair of vertices in a faulty MQ_n with up to $n-3$ faults. This result is optimal in the following sense. Assume that there are $n-2$ faults in a Möbius cube MQ_n. It is possible that there exists a vertex v with degree 2 in this faulty MQ_n. Let x and y be the two vertices adjacent to v. Then, x and y can not be the end points of any hamiltonian path since such a path must traverse both v and other vertices. We will also demonstrate that a ring of length $2^n - f_v$ can be embedded in a faulty MQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n-2$ and $n \geq 3$. All of the fault-free vertices can be included in the ring in the faulty MQ_n. In other words, we willg show that the faulty MQ_n remains hamiltonian with up to $n-2$ faults. This result is also optimal, for no regular graphs of degree n can hold over $n-2$ faults and still guarantee the existence of a fault-free hamiltonian cycle.

The rest of this paper is organized as follows. Section 2 explains the notations and the basic properties of Möbius cube. The main theorem is proved in section 3. The conclusion is given in section 4.

2 Notations and basic properties

Our fundamental graph terminologies refer to [Harary72] when using undirected graph to model interconnection networks. Given a graph, the vertex set and the edge set of G are denoted by $V(G) = V$ and $E(G) = E$, respectively. A ***path***, $P(v_0, v_t) = \langle v_0, v_1, \ldots, v_t \rangle$, is a sequence of nodes such that two consecutive nodes are *adjacent.* A path $\langle v_0, v_1, \ldots, v_t \rangle$ may contain other subpath, denoted as $\langle v_0, v_1, \ldots, v_i,\ P(v_i, v_j),\ v_j, v_{j+1}, \ldots, v_t \rangle$, where $P(v_i, v_j) = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$. A path that contains every vertex of G exactly once is called a hamiltonian path of G. A graph G is called *hamiltonian connected* if there exists a hamiltonian path between any two vertices of G. A path $\langle v_0, v_1, \ldots, v_t \rangle$ is called a *cycle* if $v_0 = v_t$ and $t \geq 3$. A cycle which visits each vertex in G exactly once is called a hamiltonian cycle. A graph that contains a hamiltonian cycle is called a *hamiltonian graph* (or simply *hamiltonian*).

The graph $G - F$ denotes the subgraph of G with node faults and/or edge faults; i.e., a faulty network, where $F \subset V(G) \bigcup E(G)$. Let k be a positive integer. A graph G is ***k-hamiltonian connected*** if $G - F$ is hamiltonian connected for any F with $|F| \leq k$. That is, there exists a hamiltonian path between any pair of vertices in a faulty network $G - F$. Similarly, a graph G is ***k-hamiltonian*** if $G - F$ is hamiltonian for any F with $|F| \leq k$.

We now introduce the definition of the Möbius cube [Cull91][Cull95].

Definition 1 *The Möbius cube, $MQ_n = (V, E)$, of dimension n has 2^n nodes. Each node is labeled by a unique n-bit binary string as its address and has connections to n other distinct nodes. The node with address $X = x_{n-1}x_{n-2}\ldots x_0$ connects to n other nodes Y_i, $0 \leq i \leq n-1$, where the address of Y_i satisfies one of the following conditions:*

(1) $Y_i = (x_{n-1}\ldots x_{i+1}\overline{x_i}\ldots x_0)$ *if* $x_{i+1} = 0$, *or*

(2) $Y_i = (x_{n-1}\ldots x_{i+1}\overline{x_i \ldots x_0})$ *if* $x_{i+1} = 1$

From the above definition, X connects to Y_i by complementing the bit x_i if $x_{i+1} = 0$, or by complementing all bits of $x_i \ldots x_0$ if $x_{i+1} = 1$. For the connection between X and Y_{n-1}, we can assume the unspecified x_n is either equal to 0 or equal to 1, which gives slightly different topologies. If we assume x_n to be 0, we call the network generated the "0-Möbius cube", denoted as 0-MQ_n, and if we assume x_n to be 1, we call the network generated the "1-Möbius cube", denoted as 1-MQ_n. The examples of 0-MQ_4 and 1-MQ_4 are shown in Figure 1. This Figure also illustrates the expansibility of the Möbius cube networks, by showing a 0-MQ_3 connects to a 1-MQ_3 to create a 0-MQ_4 and a 1-MQ_4 (the new connections are shown in dashed lines).

According to the above definition, 0-MQ_{n+1} and 1-MQ_{n+1} can be recursively constructed from a 0-MQ_n and a 1-MQ_n by adding 2^n edges. 0-MQ_{n+1} is constructed by connecting all pairs of nodes that differ only in the n-th bit, and 1-MQ_{n+1} is constructed by connecting all pairs of nodes that differ in the n-th through the 0-th bits.

For convenience, we denote MQ_{n-1}^0 and MQ_{n-1}^1 as the two subMöbius cubes of MQ_n, where MQ_{n-1}^0 (MQ_{n-1}^1 respectively) is an $(n-1)$-dimensional 0-Möbius cube (1-Möbius cube respectively) which includes all the vertices with address $0u_{n-2}\ldots u_0$ ($1u_{n-2}\ldots u_0$ respectively). In addition, we define the edge set $E_c = \{(u_0, u_1) \mid (u_0, u_1) \in E, u_0 \in MQ_{n-1}^0 \text{ and } u_1 \in MQ_{n-1}^1\}$ of MQ_n as the set of crossing edges of MQ_n. For any edge $e = (u_0, u_1) \in E_c$, the vertices u_0 and u_1 are called crossing nodes of each other. Indeed, there are 2^{n-1} crossing edges and 2^{n-1} pairs of crossing nodes in MQ_n.

3 Hamiltonian cycle in Möbius cubes

We will demonstrate that MQ_n is $(n-3)$-hamiltonian connected, for $n \geq 3$. Moreover, we will prove that a ring of length $2^n - f_v$ can be embedded in MQ_n with f_v faulty nodes and f_e faulty edges, where $f_v + f_e \leq n-2$. That is, we will prove that MQ_n

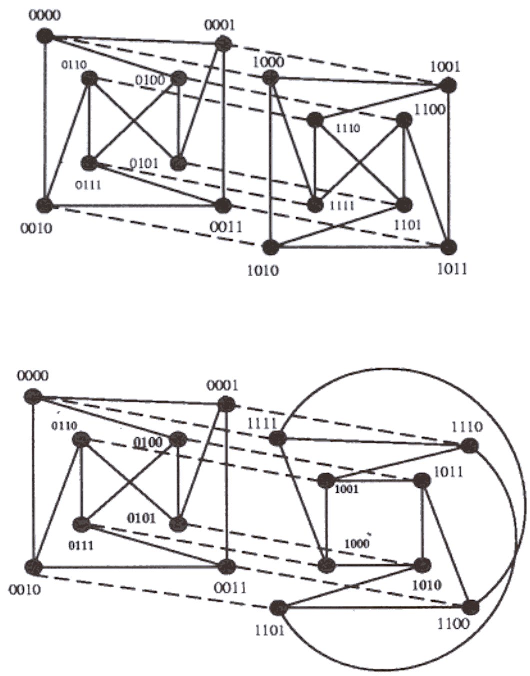


Figure 1: Examples of 0-MQ_4 and 1-MQ_4

is $(n-2)$-hamiltonian, for $n \geq 3$. We use the notation $|F| = f_e + f_v$. Our proof is by induction on n, and the outline of our proof is as follows: First, for the induction base, we prove that both 0-MQ_3 and 1-MQ_3 are hamiltonian connected and 1-hamiltonian, and both 0-MQ_4 and 1-MQ_4 are 1-hamiltonian connected and 2-hamiltonian. Next, assuming MQ_{k-1} is $(k-4)$-hamiltonian connected and $(k-3)$-hamiltonian, and MQ_k is $(k-3)$-hamiltonian connected and $(k-2)$-hamiltonian, for $4 \leq k \leq n$, we will show that MQ_{n+1} is $(n-2)$-hamiltonian connected and $(n-1)$-hamiltonian.

It is known that the Möbius cubes are vertex symmetric for $n \leq 3$ and edge symmetric for $n \leq 2$. However, in general, the Möbius cube are neither vertex symmetric nor edge symmetric [Akeers][Cull95][Fan98]. Due to the lack of symmetric property, we use computer programs to verify our induction bases: Both 0-MQ_3 and 1-MQ_3 are hamiltonian connected and 1-hamiltonian, and both 0-MQ_4 and 1-MQ_4 are 1-hamiltonian connected and 2-hamiltonian. Our computer programs simply simulate various faults in all of 0-MQ_3, 1-MQ_3, 0-MQ_4 and 1-MQ_4. There are four individual group datum in our computer simulation. Since the amount of simulation datum are too large to be included in our text, we put these four groups of results and the source programs in [Huang], where readers can find datum and detailed report.

In the following four lemmas, we show that both 0-MQ_3 and 1-MQ_3 are hamiltonian connected and 1-hamiltonian, and both 0-MQ_4 and 1-MQ_4 are 1-hamiltonian connected and 2-hamiltonian.

Lemma 1 *0-MQ_3 is hamiltonian connected and 1-hamiltonian.*

Proof: There are 8 nodes and 12 edges in 0-MQ_3. We prove this lemma by the following two steps in our computer simulation.

(1). 0-MQ_3 is hamiltonian connected: There are $C_2^8 = 28$ possible pairs of nodes. For each choice, we use exhaustive search to find a hamiltonian path between them. The results are shown in [Huang]-(a). Therefore, 0-MQ_3 is hamiltonian connected.

(2). 0-MQ_3 is 1-hamiltonian: There are two subcases in this case. (i) One node is fault: There are $C_1^8 = 8$ possible choices of faulty node. In each choice, we find a fault-free hamiltonian cycle for any node fault shown in [Huang]-(b). (ii) One edge is fault: There are $C_1^{12} = 12$ possible choices of faulty edge. For each choice, we find a fault-free hamiltonian cycle for any edge fault shown in [Huang]-(c). Hence, 0-MQ_3 is 1-hamiltonian. □

The definition of 1-MQ_3 is similar to that of 0-MQ_3. The following lemma explains that 1-MQ_3 is hamiltonian connected and 1-hamiltonian by computer simulation.

Lemma 2 *1-MQ_3 is hamiltonian connected and 1-hamiltonian.*

Proof: Although the connections of 1-MQ_3 is a little different from 0-MQ_3, they have the same number of nodes and edges. Therefore, the method of showing that 1-MQ_3 is hamiltonian connected and 1-hamiltonian is similar to that of 0-MQ_3 by computer simulation. The results are shown in [Huang]-(d), (e), (f), respectively. □

We also need the following two lemmas to support our induction steps.

Lemma 3 *0-MQ_4 is 1-hamiltonian connected and 2-hamiltonian.*

Proof: There are 16 nodes and 32 edges in 0-MQ_4. We prove the lemma by the following two steps in our computer simulation.

(1). 0-MQ_4 is 1-hamiltonian connected: There are two subcases in this case. **(i)** One nodes is faulty: Since there is one node fault in 0-MQ_4, there are $C_1^{16} * C_2^{15} =$ 1680 possible pairs of nodes. For each choice, we find a hamiltonian path between any two nodes shown in [Huang]-(g). **(ii)** One edge is faulty: Since there is one edge fault in 0-MQ_4, which has 16 nodes and 32 edges, there are $C_2^{16} * C_1^{32} = 3840$ possible choices. For each choice, we find a hamiltonian path between any two nodes shown in [Huang]-(h). Hence, 0-MQ_4 is hamiltonian connected. Hence, 0-MQ_4 is hamiltonian connected.

(2). 0-MQ_4 is 2-hamiltonian: There are three subcases. **(i)** Two nodes are faulty: There are $C_2^{16} = 120$ possible choices. In each choice, we find a fault-free hamiltonian cycle for two node faults shown in [Huang]-(i). **(ii)** Two edges are faulty: There are $C_2^{32} = 496$ possible choices. In each choice, we find a fault-free hamiltonian cycle for any two edge faults shown in [Huang]-(j). **(iii)** One node and one edge are faulty: There are $16 * 32 = 512$ possible choices. In each choice, we find a fault-free hamiltonian cycle for one node and one edge fault shown in [Huang]-(k). So, 0-MQ_4 is 2-hamiltonian. □

There are two different connections in MQ_4. The following lemma explains that 1-MQ_4 is 1-hamiltonian connected and 2-hamiltonian.

Lemma 4 *1-MQ_4 is 1-hamiltonian connected and 2-hamiltonian.*

Proof: Although the connections of 1-MQ_4 is a little different from 0-MQ_4, they have the same number of nodes and edges. So, the proving method of 1-MQ_4 being 1-hamiltonian connected and 2-hamiltonian is similar to that of 0-MQ_4 shown in [Huang]-(l), (m), (n), (o), (p), respectively. □

To continue our induction proof, for simplicity, we are not to distinguish 0-Möbius from 1-Möbius for $n \geq 5$. From now on, we use MQ_n instead of 0-MQ_n and 1-MQ_n. After proving our base cases in the previous four lemmas, we now enter the induction steps of our main results. Assuming MQ_{n-1} is $(n-4)$-hamiltonian connected and $(n-3)$-hamiltonian, and MQ_n is $(n-3)$-hamiltonian connected and $(n-2)$-hamiltonian, for some n, Lemma 6 and Lemma 7 below demonstrate that MQ_{n+1} is $(n-2)$-hamiltonian connected and $(n-1)$-hamiltonian, respectively. We need the following auxiliary lemma in Lemma 6. One may skip the proof temporarily, and come back for the proof afterwards.

Lemma 5 *Assume that MQ_{n-1} is hamiltonian connected, for some n. In a fault-free MQ_n with 4 distinct vertices w, u, x, and y, if $w \in MQ_{n-1}^0$ and $u \in MQ_{n-1}^1$, then there exists a spanning subgraph consisting of two vertex disjoint paths whose endvertices are w, x and u, y (or w, y and u, x), respectively. That is, these two disjoint paths traverse all vertices of MQ_n.*

Proof: We demonstrate this lemma by the following two cases.

Case (a): x and y are in the same subMöbius cube MQ_{n-1} of MQ_n shown in Fig. (2-a): Without loss of generality, we assume that both x and y are in MQ_{n-1}^0. Since MQ_{n-1}^0 is hamiltonian connected, there exists a hamiltonian path $HP(x,y)$ between x and y. Let a and z be the two neighboring nodes of w on $HP(x,y)$. Then, $HP(x,y) = \langle x, P(x,a), a, w, z, P(z,y), y \rangle$. And let b be the crossing node of a. Assume $b \neq u$. Then, there exists a hamiltonian path between b and u because MQ_{n-1} is hamiltonian connected. Hence, $\langle x, P(x,a), a, b, HP(b,u), u \rangle$ and $\langle w, z, P(z,y), y \rangle$ are the two disjoint paths. In the case that $b = u$, we can simply replace a with z, similar argument as above still holds.

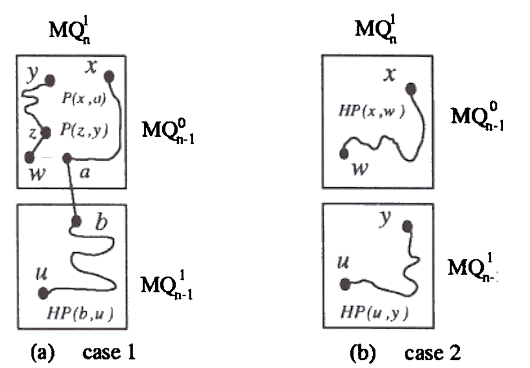


Figure 2: Illustration for Lemma 5.

Case (b): x and y are in different subMöbius cubes, say $x \in MQ_{n-1}^0$ and $y \in MQ_{n-1}^1$. Since MQ_{n-1}^0 is hamiltonian connected, there exists a hamiltonian path $HP(x,w)$ between x and w. Similarly, there exists a hamiltonian path $HP(u,y)$ between u and y. Hence, $\langle x, HP(x,w), w \rangle$ and $\langle u, HP(u,y), y \rangle$ are the two disjoint paths shown in Fig. (2-b). □

Using the result of Lemma 5, we now demonstrate that MQ_{n+1} is $(n-2)$-hamiltonian connected.

Lemma 6 *If MQ_{n-1} is $(n-4)$-hamiltonian connected and $(n-3)$-hamiltonian and MQ_n is $(n-3)$-hamiltonian connected and $(n-2)$-hamiltonian, for some n, then MQ_{n+1} is $(n-2)$-hamiltonian connected, where $n \geq 4$.*

Proof: We will show that there exists a hamiltonian path between every pair of vertices x and y in MQ_{n+1} with $|F| \leq n-2$. There are three cases: (1) all of the faults are located in the same subMöbius cube MQ_n (either $f_0 > 0, f_1 = 0, f_c = 0$ or $f_0 = 0, f_1 > 0, f_c = 0$); (2) the faults are scattered (at least two of f_0, f_1, and f_c are greater than zero); and (3) all of the faults are located in E_c ($f_0 = 0, f_1 = 0$, and $f_c > 0$).

Case 1: All of the faults are in the same subMöbius cube MQ_n.

Assume all of the faults are located in MQ_n^0. There are three subcases: (1.1) $x \in MQ_n^0$ and $y \in MQ_n^1$, (1.2) both x and y are in MQ_n^0, and (1.3) both x and y are in MQ_n^1.

Subcase (1.1). x and y are in different MQ_n^i, for $i = 0, 1$ shown in Fig. (3-a): Without loss of generality, we assume that $i = 0$ and $f_0 = n-2$. Since MQ_n^0 is $(n-2)$-hamiltonian, there exists a hamiltonian cycle $HC_0 = \langle x, u_0, P(u_0, w_0), w_0, x\rangle$ with vertices u_0 and w_0 adjacent to x. Let w_1 be the crossing node of w_0 and u_1 be the crossing node of u_0. We know that (w_0, w_1) and (u_0, u_1) are fault-free because there are no faults in E_c. Since MQ_n^1 is hamiltonian connected, there exists a hamiltonian path $HP(w_1, y)$ between w_1 and y. Hence, if $w_1 \neq y$, $\langle x, u_0, P(u_0, w_0), w_0, w_1, HP(w_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in MQ_{n+1}. Otherwise, $\langle x, w_0, P(w_0, u_0), u_0, u_1, HP(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in MQ_{n+1}.

Subcase (1.2). Both x and y are in MQ_n^0 shown in Fig. (3-b): Let d be a fault of F. Since MQ_n^0 is $(n-3)$-hamiltonian connected, $MQ_n^0 - (F - \{d\})$ contains a hamiltonian path $HP(x, y)$ between x and y. Thus, $MQ_n^0 - F$ contains two node-disjoint paths $P(x, w_0)$ and $P(u_0, y)$, where $P(x, w_0) \cup P(u_0, y) = HP(x, y) - \{d\}$. Because MQ_n is $(n-3)$-hamiltonian connected and $n - 3 \geq 0$, there exists a hamiltonian path $HP(w_1, u_1)$ between w_1 and u_1. Hence, $\langle x$, $P(x, w_0)$, w_0, w_1, $HP(w_1, u_1)$, u_1, u_0, $P(u_0, y), y\rangle$ is a hamiltonian path between x and y in MQ_{n+1}.

Subcase (1.3). Both x and y are in MQ_n^1. There are another two subcases in this case. Let $x_0 \in MQ_n^0$ be the crossing node of x and $y_0 \in MQ_n^0$ be the crossing node of y.

Subcase (1.3.1). Both x_0 and y_0 are faulty shown in Fig. (3-c): Since MQ_n^0 is $(n-2)$-hamiltonian and $f_0 = n-2$, there exists a fault-free hamiltonian cycle HC_0. Since HC_0 is a hamiltonian cycle, there are at least two edges crossing the two subMöbius cubes of MQ_n^0. Let one of the edges be (w_0, u_0) and w_1 be

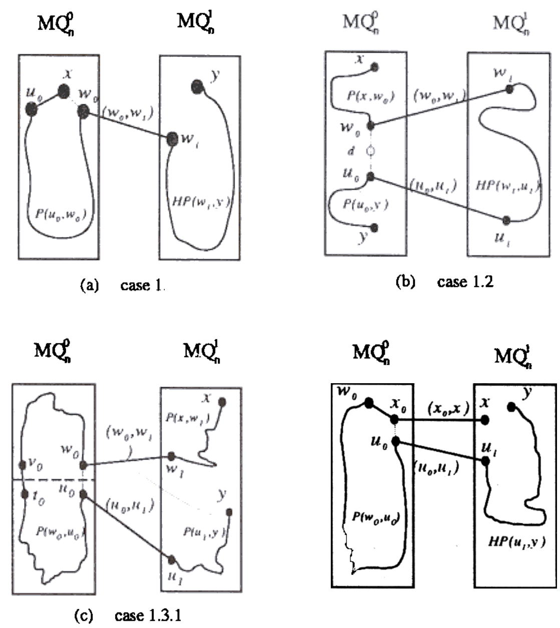


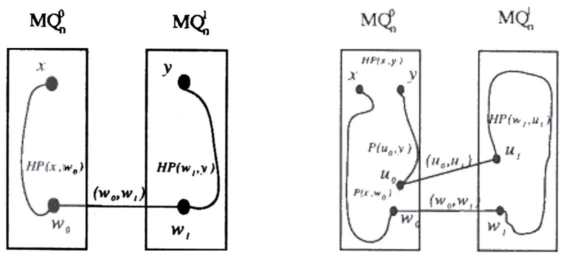
Figure 3: Illustration for Lemma 6.

the crossing node of w_0, and u_1 be the crossing node of u_0. Since w_0 and u_0 belong to different subMöbius cube of MQ_n^0, by the definition of the Möbius cube, it is not difficult to check that w_1 and u_1 must also belong to different subMöbius cube of MQ_n^1. In addition, x_0 and y_0 are both faulty, therefore $w_1 \notin \{x, y\}$ and $u_1 \notin \{x, y\}$. Since MQ_n^1 is fault-free, we have four distinct vertices u_1, w_1, x, y, and u_1, w_1 belong to different subMöbius cubes, MQ_{n-1}^0 and MQ_{n-1}^1. Therefore, by Lemma 5, there are two disjoint paths, which traverse through all vertices of MQ_n^1, say, $\langle x, P(x, w_1), w_1\rangle$ and $\langle u_1, P(u_1, y), y\rangle$. Hence, $\langle x, P(x, w_1), w_1, w_0, P(w_0, u_0)$, $u_0, u_1, P(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in MQ_{n+1}.

Subcase (1.3.2). At least one of x_0 or y_0 is fault-free shown in Fig. (3-d): Assume x_0 is fault-free. Since MQ_n^0 is $(n-2)$-hamiltonian and $f_0 = n-2$, there exists a hamiltonian cycle $HC_0 = \langle x_0, w_0$, $P(w_0, u_0)$, $u_0, x_0\rangle$ containing vertex x_0. Let u_1 be crossing node of u_0 and $u_1 \neq y$ (if $u_1 = y$, we can simply use w_0 to replace u_0). Since MQ_n^1 is $(n-3)$-hamiltonian connected and $n - 3 \geq 1$, there exists $HP(u_1, y)$ in $MQ_n^1 - \{x\}$. Hence, $\langle x, x_0, w_0$, $P(w_0, u_0)$, u_0, u_1, $HP(u_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in MQ_{n+1}.

Case 2: The faults are scattered in MQ_n^0, MQ_n^1, and E_c. Without loss of generality, we assume that $f_0 \geq f_1$. Because at least two of f_0, f_1 and f_c are greater than zero and $f_1 \leq f_0 \leq n-3$, $f_1 \leq n-3$ and $f_1 + f_c \leq n-3$, where $n \geq 4$.

There are three subcases: (2.1) $x \in MQ_n^0$ and $y \in MQ_n^1$, (2.2) both x and y are in MQ_n^0, and (2.3) both x and y are in MQ_n^1.



(a) case 2.1 (b) case 2.2

Figure 4: Illustration for Lemma 6.

Subcase (2.1). x and y are in different MQ_n^i, for $i = 0, 1$ shown in Fig. (4-a): Because there are 2^n crossing edges in MQ_{n+1}, we have at least $(2^n - (n-2)) \geq 3$ fault-free crossing edges, for $n \geq 4$. Let (w_0, w_1) be one of the fault-free crossing edges, $w_0 \neq x$, and $w_1 \neq y$. Since MQ_n^0 is $(n-3)$-hamiltonian connected and $f_0 \leq n-3$, there exist a fault-free hamiltonian path $HP(x, w_0)$ in MQ_n^0. Similarly, since MQ_n^1 is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there also exist a fault-free hamiltonian path $HP(w_1, y)$ in MQ_n^1. Hence, $\langle x, HP(x, w_0), w_0, w_1, HP(w_1, y), y\rangle$ is a fault-free hamiltonian path between x and y in MQ_{n+1}.

Subcase (2.2). Both x and y are in the same MQ_n^i, for $i = 0, 1$ shown in Fig. (4-b): Without loss of generality, we assume that $i = 0$. Since MQ_n^0 is $(n-3)$-hamiltonian connected and $f_0 \leq n-3$, there exists a hamiltonian path $HP(x, y)$ between x and y. Since $|HP(x, y)| \geq 2^n - (n-3)$, and we have at least $2^n - (n-3)$ choices, where $n \geq 4$, we can find an edge (w_0, u_0) on the path $HP(x, y)$ such that the crossing node w_1 and u_1 of w_0 and u_0, respectively, are both fault-free and the crossing edges (w_0, w_1) and (u_0, u_1) are also fault-free. Then, $HP(x, y) = \langle x, P(x, w_0), w_0, u_0, P(u_0, y), y\rangle$. Since MQ_n^1 is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there exists a hamiltonian path $HP(w_1, u_1)$ between w_1 and u_1. Hence, $(HP(x, y) \cup \{(w_0, w_1), (u_0, u_1)\} \cup HP(w_1, u_1)) - \{(w_0, u_0)\}$ is a fault-free hamiltonian path in MQ_{n+1}.

Subcase (2.3). Both x and y are in MQ_n^1: This case can be proved in a similar way to subcase (2.2).

Case 3: All of the faults are in E_c.

There are also three subcases: (3.1) $x \in MQ_n^0$ and $y \in MQ_n^1$, (3.2) both x and y are in MQ_n^0, and (3.3) both x and y are in MQ_n^1.

Subcase (3.1). $x \in MQ_n^0$ and $y \in MQ_n^1$. The conditions of this case are in fact similar to the case (2.1). The same arguments used in case (2.1) can also be applied here to obtain a fault-free hamiltonian path between x and y.

Subcase (3.2). Both x and y are in MQ_n^0. The conditions of this case are in fact similar to the case (2.2). We can find an edge (w_0, u_0) from MQ_n^0 and fault-free vertices and edges $w_1, u_1, (w_0, w_1)$, and (u_0, u_1) from MQ_n^1 and E_c with the fact that $(2^n - 2(n-2)) \geq 2$, where $n \geq 4$. Therefore, a similar hamiltonian path between x and y as in the case (2.2) can be found.

Subcase (3.3). This case can be proved in a similar way to subcase (3.2).

This completes the induction proof of Lemma 6. □

After proving MQ_{n+1} is $(n-2)$-hamiltonian connected, we now demonstrate that MQ_{n+1} is $(n-1)$-hamiltonian.

Lemma 7 *If MQ_n is $(n-3)$-hamiltonian connected and $(n-2)$-hamiltonian, for some n, then MQ_{n+1} is $(n-1)$-hamiltonian, where $n \geq 4$.*

Proof: Let E_c be the set of crossing edges; i.e., $E_c = \{(u_0, u_1) \mid (u_0, u_1) \in E, u_0 \in MQ_n^0 \text{ and } u_1 \in MQ_n^1\}$. Let F be a faulty set of MQ_{n+1} with $F_0 = F \cap MQ_n^0$, $F_1 = F \cap MQ_n^1$, and $F_c = F \cap E_c$, and let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_c = |F_c|$. We will show that MQ_{n+1} is $(n-1)$-hamiltonian in the following three cases: (1) all of the faults are located in the same subMöbius cube MQ_n (either $f_0 > 0, f_1 = 0, f_c = 0$ or $f_0 = 0, f_1 > 0, f_c = 0$); (2) the faults are scattered (at least two of f_0, f_1, and f_c are greater then zero); (3) all of the faults are located in E_c ($f_0 = 0, f_1 = 0, f_c > 0$).

Case 1: All of the faults are located in the same MQ_n^i, for $i = 0, 1$ shown in Fig. (5-a): Without loss of generality, we assume that all of the faults are located in MQ_n^0 and $f_0 = n-1$. Since MQ_n^0 is $(n-2)$-hamiltonian, there exist two vertices w_0 and u_0 such that there is a hamiltonian path $HP(w_0, u_0)$ between w_0 and u_0. Let w_1 be the crossing node of w_0 and u_1 be the crossing node of u_0. We know that $w_1, u_1, (w_0, w_1)$ and (u_0, u_1) are all fault-free because there are no faults in either E_c or MQ_n^1. Furthermore, since MQ_n^1 is hamiltonian connected, there exists a hamiltonian path $HP(u_1, w_1)$ between u_1 and w_1. Hence, $\langle w_0, HP(w_0, u_0), u_0, u_1, HP(u_1, w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle in MQ_{n+1}.

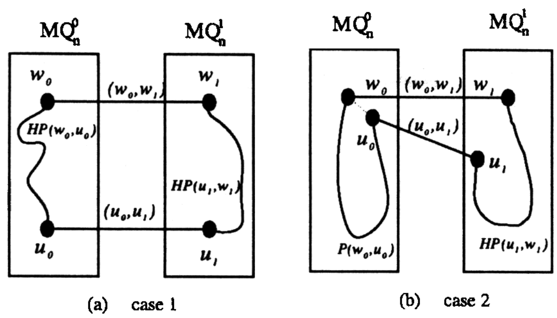


Figure 5: Illustration for Lemma 7.

Case 2: The faults are scattered in MQ^0_n, MQ^1_n, and E_c shown in Fig. (5-b): Without loss of generality, we assume that $f_0 \geq f_1$. Because at least two of f_0, f_1 and f_c are greater than zero, then $f_1 \leq f_0 \leq n-2$. We want to prove that $f_1 \leq n-3$. We know that f_1 is either strictly less than $n-2$ or equal to $n-2$. Suppose $f_1 = n-2$, then $f_0 = 1$. Since $f_0 \geq f_1$, then $1 \geq n-2$ and $3 \geq n$ contradicting the fact that $n \geq 4$. Thus, $f_1 \leq n-3$ and $f_1 + f_c \leq n-2$, where $n \geq 4$. Since MQ^0_n is $(n-2)$-hamiltonian and $f_0 \leq n-2$, there exists a hamiltonian cycle HC_0 with at least $2^n - (n-2)$ edges. We now show that there exists an edge $(w_0, u_0) \in HC_0$ such that the crossing nodes w_1 and u_1 of w_0 and u_0, respectively, are both fault-free and the crossing edges (w_0, w_1) and (u_0, u_1) are also fault-free. Since $|HC_0| \geq 2^n - (n-2)$, we have at least $2^n - (n-2)$ choices. If none of the edges of HC_0 meets the requirements of (w_0, u_0), then there are at least $\lceil \frac{2^n-(n-2)}{2} \rceil$ faults in F_1 and F_c because a single fault in either F_1 or F_c eliminates at most 2 edges of HC_0 contradicting the fact that $f_1 + f_c \leq n-2$, for $n \geq 4$. Therefore, we can find such an edge (w_0, u_0) and then $HC_0 = \langle w_0, P(w_0, u_0), u_0, w_0\rangle$. Because MQ^1_n is $(n-3)$-hamiltonian connected and $f_1 \leq n-3$, there exists a hamiltonian path between u_1 and w_1, i.e., $HP(u_1, w_1)$. Hence, $\langle w_0, P(w_0, u_0), u_0, u_1, HP(u_1, w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle between x and y in MQ_{n+1}.

Case 3: All of the faults are in E_c. Because there are 2^n crossing edges in MQ_{n+1}, there are at least $(2^n - (n-1)) \geq 2$ fault-free crossing edges, where $n \geq 4$. We can choose two fault-free crossing edges (w_0, w_1) and (u_0, u_1). Since both MQ^0_n and MQ^1_n are $(n-3)$-hamiltonian connected, there exist $HP(w_0, u_0)$ in MQ^0_n and $HP(u_1, w_1)$ in MQ^1_n. Hence, $\langle w_0, HP(w_0, u_0), u_0, u_1, HP(u_1, w_1), w_1, w_0\rangle$ is a fault-free hamiltonian cycle in MQ_{n+1}.

This completes the induction proof of Lemma 7. □

After proving both the induction bases and the induction steps, now we are ready to prove our main theorem.

Theorem 1 *MQ_n is (n-3)-hamiltonian connected and (n-2)-hamiltonian, for $n \geq 3$.*

Proof: By both Lemma 1 and Lemma 2, 0-MQ_3 and 1-MQ_3 are hamiltonian connected and 1-hamiltonian, and both Lemma 3 and Lemma 4, 0-MQ_4 and 1-MQ_4 are 1-hamiltonian connected and 2-hamiltonian. Then, by Lemma 6 and Lemma 7, and by a simple induction, MQ_n is (n-3)-hamiltonian connected and (n-2)-hamiltonian, for all $n \geq 3$. □

4 Conclusions

This paper focuses on the study of a faulty Möbius n-cube, $MQ_n - (f_v + f_e)$, with f_v faulty nodes and f_e faulty edges. We have proved two optimal results: There exists a hamiltonian path between any pair of vertices in a faulty MQ_n with up to $n-3$ faults; a ring of length $2^n - f_v$ can be embedded in a faulty MQ_n with $f_v + f_e \leq n-2$. Many other topological properties of the Möbius cube have been explored as in [Cull91][Fan98] recently. They demonstrated that some properties and performance of Möbius cube are better than those of the hypercubes. Therefore, the Möbius cube can be considered as an attractive alternative to the hypercube.

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