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An Extension of the ELECTRE III Method based on the 2-tuple Linguistic Representation Model for Dealing with Heterogeneous Information

Juan Carlos Leyva-López[1,*], Jesús Jaime Solano-Noriega[2],
Jorge Anselmo Rodríguez-Castro[1], Pedro J. Sánchez[3]

[1] Universidad Autónoma de Occidente, Culiacán,
México

[2] Universidad Autónoma de Occidente, Los Mochis,
México

[3] Universidad de Jaén,
Spain

{juan.leyva, jaime.solano, jorge.rodriguez}@uadeo.mx, pedroj@ujaen.es

Abstract. Multicriteria decision analysis (MCDA) is a problem-solving approach that helps to tackle complex decision-making problems. It involves analyzing a set of alternatives that are assessed based on a set of decision criteria by one or more decision-makers. These decision-makers use both subjective and objective judgments, which can be qualitative and/or quantitative. The goal of MCDA is to arrive at a decision that is fair, effective, and considers all relevant factors. Some MCDA methods lack mechanisms to consistently process heterogeneous information provided by the decision-maker and reduce it simplistically to numerical values to assess subjective criteria and thus obtain numerical results with low interpretability. This paper presents an extension of the ELECTRE III method that considers heterogeneous information provided by the decision-maker as input data in the decision criteria. The new proposal is based on the 2-tuple linguistic representation model, which allows for a flexible assessment structure in which the decision-maker can provide their preferences by applying diverse levels of information according to the nature and uncertainty of the decision criteria. It includes a new distance measure based on linguistic transformations appropriate for the MCDA outranking approach. Finally, the viability and pertinence of the proposed method are shown in a case to evaluate the environmental impact that can occur between the interactions of some industrial activities in a petrol station.

Keywords. Computing with words, heterogeneous information, linguistic preferences, multicriteria decision analysis, ELECTRE III.

1 Introduction

Multicriteria Decision Analysis (MCDA) provides a methodological framework for managing complex decision-making problems with multiple criteria in conflict. The purpose and scope of MCDA are to support decision-makers (DMs) while addressing complex decision-making problems.

The MCDA outranking approach involves ranking a set of alternatives in decreasing order of preferences. This is a multicriteria ranking problem, where there may be ties and incomparability among the alternatives [1, 2]. The ranking means a recommendation for the DM generated by the solution method.

The MCDA outranking methods combine the aggregation and exploitation phases. In the preference aggregation phase, the DM's preference aggregation model is achieved, represented by an outranking relation, which usually does not present attractive mathematical

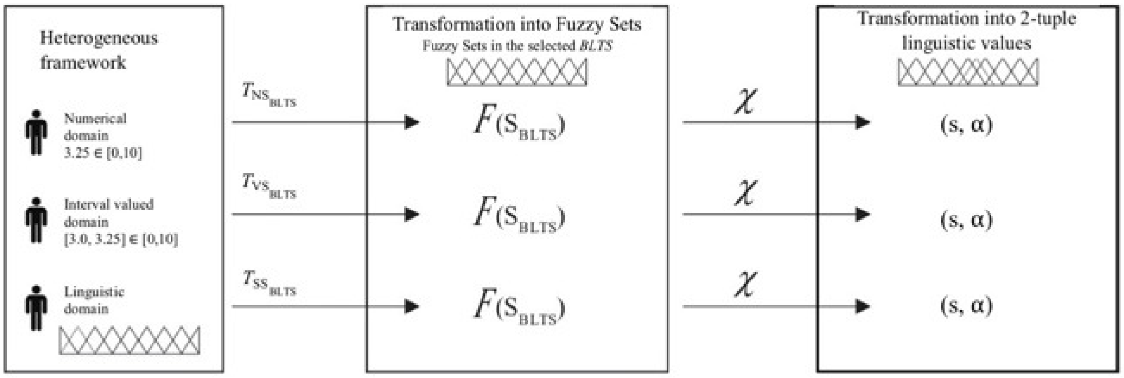

Fig. 1. Fusion approach dealing with heterogeneous information using the 2-tuple fuzzy linguistic model

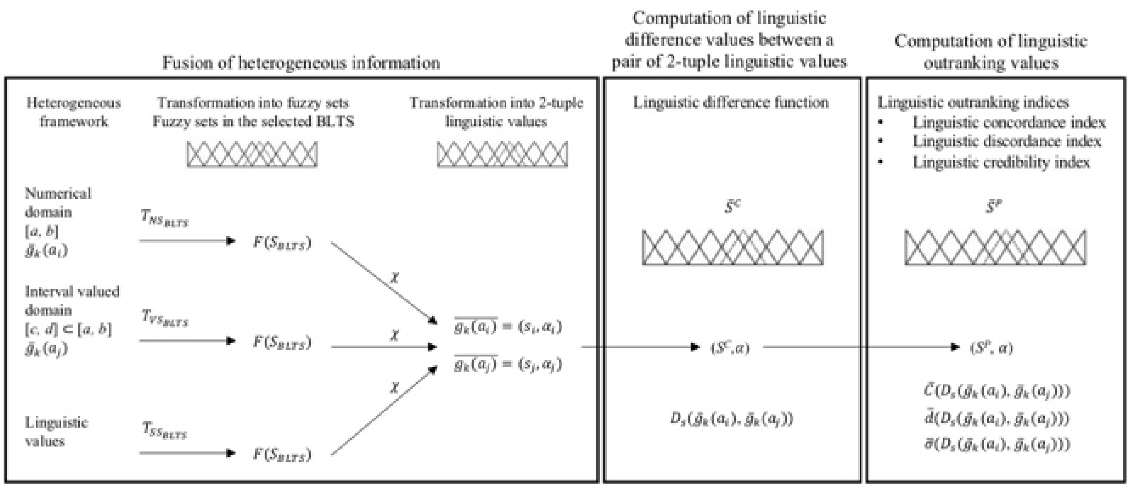

Fig. 2. Process for modeling linguistic outranking indices

properties such as transitivity and completeness [3].

In the exploitation phase, a partial preorder is deduced from the outranking relation, reflecting irreducible incomparability and indifference between the alternatives [4].

ELECTRE III [5] is a representative method of MCDA that constructs and exploits a fuzzy outranking relation.

The ELECTRE III method has been widely used to solve many real-world problems that can be formulated as multicriteria ranking problems. However, like any multicriteria method, it has weaknesses and limitations when using it for specific instances of the ranking problem. For example, ELECTRE III is inadequate regarding:

– Dealing with heterogeneous information: DMs must use numerical scales in ELECTRE III, which is inflexible as criteria can have varied descriptions and be assessed in different expression domains.
– Dealing with uncertainty: ELECTRE III cannot correctly handle the uncertainties and vagueness of subjective judgments.

This paper proposes an extension of the ELECTRE III method to reduce limitations in information management, incorporating a flexible heterogeneous evaluation structure in the decision criteria to analyze elements of uncertainty and vagueness that occur in many instances of the multicriteria ranking problem, which is more in line

Table 1. Criteria set description

	Name	Description	Expression Domain
g_1	Intensity	How the action impacts the factor	Linguistic: L
g_2	Extension	The range within which the action affects the site	Linguistic: L
g_3	Moment	The duration from the action's onset to the time the factor begins to be affected	Numerical: N
g_4	Persistence	The estimated duration of the action's effect	Valor Interval-valued: I
g_5	Reversibility	The potential for the factor to be naturally restored to its original state after being affected	Numerical: N
g_6	Synergy	The strengthening of straightforward impacts	Linguistic: L
g_7	Accumulation	The gradual escalation in the expression of the effect	Linguistic: L
g_8	Effect	How the action's impact on an environmental factor becomes apparent	Linguistic: L
g_9	Recoverability	The potential for the factor to be restored with the help of human action	Linguistic: L
g_{10}	Periodicity	The consistency in which the impact on the environmental factor is observed	Linguistic: L

Table 2. Assessment for each criterion on each *EI* using a heterogeneous framework

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
ei_1	L	L	0,00	[0,0.2]	1,00	L	L	L	H	VL
ei_2	H	VL	0,00	[0,0.2]	1,00	M	M	M	VH	VL
ei_3	H	L	0,00	[0.4,0.6]	10,00	M	L	VH	L	M
ei_4	M	VL	0,10	[0,0.2]	2,00	L	VL	VL	M	H
ei_5	H	L	0,10	[0,0.05]	10,00	M	M	VH	L	M
ei_6	VH	L	0,00	[0.8,1.0]	10,00	M	M	VH	L	H
ei_7	VL	L	1,50	[0.8,1.0]	10,00	VL	L	VL	L	VH

with the quantitative and qualitative essence of the decision criteria and with the experience of the DM.

The fusion linguistic approach converts heterogeneous information into a linguistic one [4, 5, 6]. The fusion approach for an MCDA method makes the computations possible and generates interpretable results [2]; however, it implies the need for utilizing Computing with Words (CW) procedures [7, 8].

Therefore, we employ the fuzzy linguistic approach based on the 2-tuple linguistic representation model [9, 10] and a linguistic-based distance measure [11, 12] to construct a fuzzy outranking relation in such a way that the DM can make his evaluations in different sets of linguistic terms according to his knowledge of the decision problem [6].

We consider a modified distillation procedure to derive a partial preorder of the alternatives for the exploitation phase of the fuzzy outranking relation based on 2-tuple linguistic representation modeling. This approach's main advantage is tackling the uncertainty of criteria performances and DMs' knowledge without losing information.

Thus, this new method will be helpful in multicriteria ranking problems whose DMs express their value judgments through heterogeneous values. In these kinds of issues, it is common for

Table 3. Fused information supplied by the DM

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
ei_1	(L,0)	(L,0)	(VH,0)	(VL,0.44)	(VL,0.4)	(L,0)	(L,0)	(L,0)	(H,0)	(VL,0)
ei_2	(H,0)	(VL,0)	(VH,0)	(VL,0.44)	(VL,0.4)	(M,0)	(M,0)	(M,0)	(VH,0)	(VL,0)
ei_3	(H,0)	(L,0)	(VH,0)	(M,0)	(VH,0)	(M,0)	(L,0)	(VH,0)	(L,0)	(M,0)
ei_4	(M,0)	(VL,0)	(VH,-0.04)	(VL,0.44)	(L,-0.2)	(L,0)	(VL,0)	(VL,0)	(M,0)	(H,0)
ei_5	(H,0)	(L,0)	(VH,-0.04)	(VL,0.17)	(VH,0)	(M,0)	(M,0)	(VH,0)	(L,0)	(M,0)
ei_6	(VH,0)	(L,0)	(VH,0)	(VH,-0.44)	(VH,0)	(M,0)	(M,0)	(VH,0)	(L,0)	(H,0)
ei_7	(VL,0)	(L,0)	(H,0.4)	(VH,-0.44)	(VH,0)	(VL,0)	(L,0)	(VL,0)	(L,0)	(VH,0)

DMs to have different levels of knowledge and domains of the criteria.The organization for the rest of the document is presented as follows: Section 2 shows background about ELECTRE III and linguistic and heterogeneous information. Then, section 3 gives the linguistic ELECTRE III, and section 4 provides an illustrative example. Finally, section 5 points out the conclusions.

2 Background

This section briefly describes the ELECTRE III method and the use of linguistic and heterogeneous information in MCDA.

2.1 The ELECTRE III Method

The ELECTRE III method is a decision support technique under the category of outranking methods designed to solve problems involving multiple criteria [13, 14]. It is a multicriteria ranking method that is relatively simple in conception and application compared to other MCDA methods. It can be applied to situations where a finite set of alternatives must be prioritized, considering multiple, often contradictory, criteria (e.g. [15, 16]).

ELECTRE III needs a decision matrix with criteria evaluations for each alternative and preference information in the form of weights and thresholds. The model accounts for uncertainty in the assessments when defining the thresholds.

This method consists of two distinct stages. Initially, (*i*) it aggregates the input data to create a fuzzy outranking relation on the pairs of alternatives.

Subsequently, (*ii*) it exploits the fuzzy outranking relation to generate a partial preorder of alternatives [14]. Let us have the following notations:

$A = \{a_1, a_2, \dots, a_m\}$ is defined as the set of potential decision alternatives.

$G = \{g_1, g_2, \dots, g_n\}$ is defined as a coherent family of criteria.

$W = \{w_1, w_2, \dots, w_n\}$ is defined as the set of weights that reflects the DM's preferences over the set G. Let us assume that $\sum_{i=1}^{n} w_i = 1$.

$g_j(a_i)$ is the evaluation of the criterion g_j for alternative a_i.

The ELECTRE III method uses the fundamental principle of threshold values; an indifferent q threshold is determined as:

$$a_i P a_j \ (a_i \ is \ prefered \ to \ a_j) \leftrightarrow g(a_i) > g(a_j) + q.$$

And

$$a_i I a_j \ (a_i \ is \ indifferent \ to \ a_j) \leftrightarrow |g(a_i) - g(a_j)| \leq q.$$

The inclusion of the indifference threshold addresses the consideration of a DM's sentiment toward practical comparisons of the alternatives. However, a marker still exists when a DM's preference transitions from a state of indifference to a state of strict preference. Regarding conceptual understanding, it is beneficial to introduce a buffer region between indifference and strict preference.

Table 4. Indifference q preference p and veto v values

Criterion	$\left(\left(I^{C}_{t_{q_k}}, \alpha_{q_k}\right) q\right)$	$\left(\left(I^{C}_{t_{p_k}}, \alpha_{p_k}\right) p\right)$	$\left(\left(I^{C}_{t_{v_k}}, \alpha_{v_k}\right) v\right)$
g_1	$(I^C_5, 0)q$	$(I^C_6, 0)p$	$(I^C_7, 0)v$
g_2	$(I^C_5, 0)q$	$(I^C_6, 0)p$	$(I^C_7, 0)v$
g_3	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_4	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_5	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_6	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_7	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_8	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_9	$(I^C_5, 0)q$	$(I^C_6, 0)p$	
g_{10}	$(I^C_5, 0)q$	$(I^C_6, 0)p$	

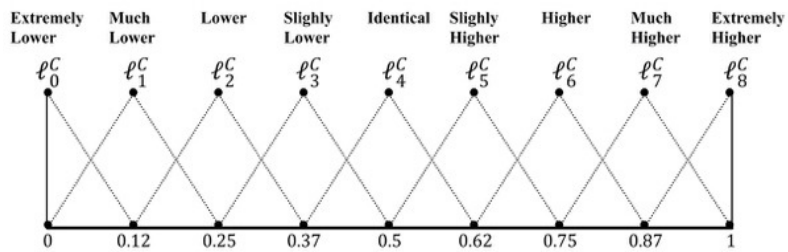

Fig. 3. Linguistic comparison scale S^C

This intermediary region represents a state where the DM hesitates over indifference and preference, known as a weak preference. Like the preference relations indifference (I) and strict preference (P), this zone of hesitation is modeled by introducing a preference threshold, denoted as p. Therefore, we adopt an indifference-preference model, incorporating a binary relation Q for weak preference measurement:

$$\begin{gathered} a_iPa_j \left(a_i \textit{ is strictly preferred to } a_j\right) \leftrightarrow g(a_i) - g(a_j) > p \\ a_iQa_j \left(a_i \textit{ is weakly preferred to } a_j\right) \leftrightarrow q < g(a_i) - g(a_j) \\ \leq p \\ a_iIa_j \left(a_i \textit{ is indifferent to } a_j; \textit{and } a_j \textit{ to } a_i\right) \\ \leftrightarrow \left|g(a_i) - g(a_j)\right| \leq q \end{gathered} \tag{1}$$

The selection of thresholds significantly impacts the determination of specific binary relations. In [14, 17], detailed information is provided on how to compute thresholds in ELECTRE III, including their nature, meaning, and form.

We must acknowledge that we have solely examined the basic scenario where the thresholds q and p are constant values rather than functions dependent on the criteria's values. The ELECTRE method can be presented in a more straightforward way by using constant thresholds. However, employing variable thresholds in situations where criteria with higher values could result in more substantial indifference and preference thresholds might be advantageous.

ELECTRE III uses these thresholds in the aggregation procedure to create the outranking relation O. Based on the DM' preference model, we can justify that "a_i is at least as good as a_j "denoted as a_iOa_j. Subsequently, each pair of alternatives is evaluated to verify the validity of a_iOa_j from which one of the following states can happen:

i) a_iOa_j and $\neg(a_jOa_i)$; ii) $\neg(a_iOa_j)$ and a_jOa_i; iii) a_iOa_j and a_jOa_i; and iv) $\neg(a_iOa_j)$ and $\neg(a_jOa_i)$. States iii and iv agree with the indifference and incomparability preference relations denoted as I and R *respectively*.

There are two principles that ELECTRE III incorporates to validate a_iOa_j, the concordance and the non-discordance principles. The former holds that most criteria, considering their respective significance, support the assertion $a_i O a_j$; meanwhile, the second principle holds that a minority of the criteria are against $a_i O a_j$. These two principles are executed as follows: suppose criteria are to maximize; first, we examine the outranking relation established for each criterion where $a_i O_k a_j$ denotes "a_i is at least as good as a_j" on criterion $k: k = 1,2,\ldots,n.$

Let q and p be an indifference and preference thresholds, in this scenario, criterion k is in concordance with $a_i O a_j$ iif $g_k(a_i) \geq g_k(a_j) - q_k$ (e.g., $a_i O_k a_j$). Conversely, criterion k is in discordance with $a_i O_k a_j$ iif $g_k(a_j) \geq g_k(a_i) + p_k$ (e.g. $a_j P_k a_i$). Hence, using the concordance and discordance rules, the claim $a_i O_k a_j$ can be assessed.

The initial stage involves creating a concordance assessment, represented by the concordance index $C(a_i, a_j)$ for each $(a_i, a_j) \in A \times A$. Suppose that w_k represents the weight for the $k-th$ criterion, the concordance index (Eq. 2) can be expressed as follows:

$$C(a_i, a_j) = \frac{1}{W} \sum_{k=1}^{n} w_k c_k(a_i, a_j), \tag{2}$$

where:

$$W = \sum_{k=1}^{n} w_k. \tag{3}$$

And:

$$c_k(a_i, a_j) = \begin{cases} 0, if\ g_k(a_i) + p_k \leq g_k(a_j), \\ \dfrac{p_k + g_k(a_i) - g_k(a_j)}{p_k - q_k}, \\ if\ g_k(a_j) - p_k \leq g_k(a_i) < g_k(a_j) - q_k, \\ 1, if\ g_k(a_i) + q_k \geq g_k(a_j), \end{cases} \tag{4}$$

where k = 1,2,…,n.

The second stage involves creating the discordance index, which integrates the veto v threshold. With this threshold, it is possible to reject completely $a_i O a_j$ if, for any given criterion veto threshold, v_k, $g_k(a_j) > g_k(a_i) + v_k$. The calculation of the discordance index $d_k(a_i, a_j)$ (Eq. 5) for each criterion k is performed as follows:

$$d_k(a_i, a_j) = \begin{cases} 0, if\ g_k(a_i) + p_k \geq g_k(a_j), \\ \dfrac{g_k(a_j) - g_k(a_j) - p_k}{v_k - p_k}, \\ if \\ g_k(a_j) - v_k < g_k(a_i) < g_k(a_j) - p_k \\ 1, if\ g_k(a_i) + v_k \leq g_k(a_j), \end{cases}, \tag{5}$$

where k=1,2 ,..., n.

Finally, both measures, concordance, and discordance, must be fused to make a metric that reflects the power of the affirmation $a_i S a_j$. This metric is known as the credibility index $\sigma(a_i, a_j)(0 \leq \sigma(a_i, a_j) \leq 1)$ and is stated in (Eq. 6) as:

$$\sigma(a_i, a_j) = \begin{cases} C(a_i, a_j), if\ K(a_i, a_j) = \emptyset \\ C(a_i, a_j) \times \\ \prod_{k \in K(a_i, a_j)} \dfrac{1 - d_k(a_i, a_j)}{1 - C(a_i, a_j)}, \\ if\ K(a_i, a_j) \neq \emptyset, \end{cases} \tag{6}$$

where $K(a_i, a_j)$ contains the criteria such that $d_k(a_i, a_j) > C(a_i, a_j)$.

Equation 6 operates under the idea that if the magnitude of the concordance exceeds that of the discordance, there is no need to alter the concordance value. However, if this condition is not met, we must question the assertion $a_i S a_j$ and adjust the value of $C(a_i, a_j)$ accordingly.

In the scenario where the discordance value is 1.0 for any $(a_i, a_j) \in A \times A$ on any criterion k, there is no assurance that $a_i S a_j$, hence the outranking degree is $\sigma(a_i, a_j) = 0.0$.

Based on Eq. 6, we can create a fuzzy outranking relation O_A^σ stated on $A \times A$ where any $(a_i, a_j) \in A \times A$ has a value $\sigma(a_i, a_j), (0 \leq \sigma(a_i, a_j) \leq 1)$ indicating the power of $a_i O a_j$.Once the model is complete, the subsequent phase in the approach involves exploiting the outranking model O_A^σ to generate a ranking of alternatives. ELECTRE III employs the distillation algorithm [17] to exploit the outranking model O_A^σ to produce a ranking.

However, due to space limitations, we will not elaborate on the details of this procedure here.

Table 5. Concordance indices on the criterion g_1

	ei_1	ei_2	ei_3	ei_4	ei_5	ei_6	ei_7
ei_1	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_4^P,0)$	$(S_8^P,0)$
ei_2	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_3	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_4	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_5	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_6	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_7	$(S_8^P,0)$	$(S_4^P,0)$	$(S_4^P,0)$	$(S_8^P,0)$	$(S_4^P,0)$	$(S_4^P,0)$	$(S_8^P,0)$

Table 6. Comprehensive linguistic concordance matrix $\bar{C}(ei_i, ei_j)$

$\bar{C}(ei_i, ei_j)$	ei_1	ei_2	ei_3	ei_4	ei_5	ei_6	ei_7
ei_1	$(S_8^P,0)$	$(S_8^P,0)$	$(S_7^P,0.584)$	$(S_7^P,0.84)$	$(S_7^P,0.584)$	$(S_5^P,0.8048)$	$(S_7^P,0.2448)$
ei_2	$(S_8^P,0)$	$(S_8^P,0)$	$(S_7^P,0.744)$	$(S_7^P,0.84)$	$(S_7^P,0.744)$	$(S_7^P,0.4048)$	$(S_7^P,0.2448)$
ei_3	$(S_8^P,0)$	$(S_7^P,0.68)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_4	$(S_8^P,0)$	$(S_8^P,0)$	$(S_7^P,0.488)$	$(S_8^P,0)$	$(S_7^P,0.488)$	$(S_7^P,0.3088)$	$(S_7^P,0.6288)$
ei_5	$(S_8^P,0)$	$(S_7^P,0.68)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_7^P,0.7776)$	$(S_7^P,0.7776)$
ei_6	$(S_8^P,0)$	$(S_7^P,0.68)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$	$(S_8^P,0)$
ei_7	$(S_8^P,0)$	$(S_6^P,0.24)$	$(S_6^P,0.24)$	$(S_8^P,0)$	$(S_6^P,0.24)$	$(S_4^P,0.8)$	$(S_8^P,0)$

Instead, in the following subsections, we introduce basic concepts of computers with words.

2.2 Linguistic Information and Management of Heterogeneous Information in MCDA

This section provides an overview of approaches to handling the three types of information in the heterogeneous framework. It introduces the 2-tuple linguistic representation model, which is appropriate for our problem because it enhances the interpretability of the MCDA process, which are the main required features of our proposal.

2.2.1 The Heterogeneous Framework

Here, the evaluation framework calculates a global evaluation that condenses the gathered information and gives helpful decision-making results.

The DM can naturally declare his preferences in different information domains and obtain a heterogeneous structure [18]. The following expression domains are used in the linguistic extension of the ELECTRE III method:

– Numerical values (N): $g_j(a_i) = v_{ij} \in [a,b], a, b \in \Re$ Represents assessments related to quantitative criteria.

– Interval values (V): $g_j(a_i) = V([a,b]) = [a_{ij}, b_{ij}] \in [a,b]$, and $a_{ij} \leq b_{ij}$. When exact numbers are unavailable, decision-makers use imprecise quantitative criteria.

– Linguistic values (S): $g_j(a_i) = s_{ij} \in S$, $S = \{s_0, \ldots, s_h\}$, being $h+1$ the number of elements of the linguistic term set (LTS) S.

Assessing qualitative criteria is familiar to them. The linguistic approach is appropriate for representing data through linguistic variables. [19, 20].

2.2.2 The 2-tuple Linguistic Representation Model

Handling heterogeneous information can be done using processes based on computing with words [20]. The models most frequently used for the treatment of heterogeneous information are:

i. The semantic model utilizes linguistic terms as labels to represent fuzzy numbers, while the computations are performed directly on the fuzzy numbers.

ii. The symbolic model that utilizes an order index of the linguistic terms to perform direct calculations on the labels.

Table 7. Linguistic discordance matrix on a criterion g_1

	ei_1	ei_2	ei_3	ei_4	ei_5	ei_6	ei_7
ei_1	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_2	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_3	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_4	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_5	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_6	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$
ei_7	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$	$(S_0^P, 0)$

Table 8. Linguistic credibility matrix in the linguistic extension of the ELECTRE III

$\bar{C}(ei_i, ei_j)$	ei_1	ei_2	ei_3	ei_4	ei_5	ei_6	ei_7
ei_1	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_7^P, 0.584)$	$(S_7^P, 0.84)$	$(S_7^P, 0.584)$	$(S_5^P, 0.8048)$	$(S_7^P, 0.2448)$
ei_2	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_7^P, 0.744)$	$(S_7^P, 0.84)$	$(S_7^P, 0.744)$	$(S_7^P, 0.4048)$	$(S_7^P, 0.2448)$
ei_3	$(S_8^P, 0)$	$(S_7^P, 0.68)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$
ei_4	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_7^P, 0.488)$	$(S_8^P, 0)$	$(S_7^P, 0.488)$	$(S_7^P, 0.3088)$	$(S_7^P, 0.6288)$
ei_5	$(S_8^P, 0)$	$(S_7^P, 0.68)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_7^P, 0.7776)$	$(S_7^P, 0.7776)$
ei_6	$(S_8^P, 0)$	$(S_7^P, 0.68)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$	$(S_8^P, 0)$
ei_7	$(S_8^P, 0)$	$(S_6^P, 0.24)$	$(S_6^P, 0.24)$	$(S_8^P, 0)$	$(S_6^P, 0.24)$	$(S_4^P, 0.8)$	$(S_8^P, 0)$

This research uses the symbolic model to calculate the linguistic evaluations in the ELECTRE III method, using the 2-tuple linguistic representation model developed by [11]. In the rest of this section, we present the basics of the 2-tuple linguistic representation model.

Definition 2.1. [4]. Let $\mathrm{S} = \{\mathrm{s}_0, \ldots, \mathrm{s}_\mathrm{h}\}$ be a linguistic term set. The symbolic translation of a linguistic term $\mathrm{s_i} \in \mathrm{S} = \{\mathrm{s}_0, \ldots, \mathrm{s}_\mathrm{h}\}$ is a numerical value α assessed in $[-0.5, 0.5)$ that supports the “difference of information” between an amount of information $\beta \in [0, \mathrm{h}]$ and the closest value in $\{0, \ldots, \mathrm{h}\}$ that specifies the index of the closest linguistic term in $\mathrm{S}(\mathrm{s_i})$ being $[0, \mathrm{h}]$ the interval of granularity of S.

Based on this meaning, a linguistic representation model must be built that denotes linguistic information using a 2-tuple (s_i, α_i), $s_i \in S$ means the linguistic label of information and $\alpha_i \in [-0.5, 0.5)$ is a numerical value stating the translation starting from the original result β with the index nearest to the label, i, in the linguistic term set $S(s_i)$, i.e., the symbolic translation. Moreover, this model states transformation functions between the numerical values and the linguistic 2-tuple.

Definition 2.2. [4]. Let $S = \{s_0, \ldots, s_h\}$ a linguistic term set and $\beta \in [0, h]$ a value supporting the result of a symbolic aggregation operation, then the 2-tuple that states the equivalent information to β is calculated with the following function:

$$\Delta_\mathrm{S}: [0, \mathrm{h}] \rightarrow \mathrm{S} \times (-0.5, .0.5), \tag{7}$$

$$\Delta_S(\beta) = (s_i, \alpha), with \quad \begin{cases} s_i & i = round(\beta), \\ \alpha = \beta - i & \alpha \in [-0.5, 0, 5), \end{cases} \tag{8}$$

where round (·) is the typical round operation, S_i has the closest index label to β, and α is the value of the symbolic translation.

Let $S = \{s_0, \ldots, s_h\}$ be a linguistic term set and (s_i, α_i) be a linguistic 2-tuple. From (7) and (8), a ${\Delta_S}^{-1}$ function can be defined, so that, from a 2-tuple (s_i, α_i), ${\Delta_S}^{-1}$ returns its equivalent numerical value $\beta \in [0, h]$ in the interval of granularity of S as follows:

$${\Delta_S}^{-1}: S \times -0{,}5{,}0.5) \rightarrow [0, h], \tag{9}$$

$${\Delta_S}^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{10}$$

Note that to transform a linguistic term into a linguistic 2-tuple, append a value 0 as symbolic translation: $s_i \in S \Rightarrow (s_i, 0)$.

Example 1. Let us suppose a symbolic aggregation operator, φ(.) whose input are different labels assessed in *S* = {*nothing*; *very low*; *low*; *medium*; *high*; *very high*; *perfect*}, obtaining the following results:

Table 9. Crisp outranking relation

ei_iOei_j	ei_1	ei_2	ei_3
ei_1	0	0	0
ei_2	0	0	0
ei_3	1	0	0
ei_4	1	1	0
ei_5	1	0	0
ei_6	1	0	0
ei_7	1	0	0

φ (medium; medium; medium; very high) = 3:21 = β_1

φ (low; medium; very low; high) = 2:76 = β_2

Being β_1=3:21 and β_2=2:76, then the 2-tuple linguistic values (Definition 2.2) of these symbolic results, which do not match with any linguistic term in *S,* are:

Δ_S(3:21) = (s_3;0:21) = (*medium*, 0.21).

The symbolic translation (definition 2.1) α is 0.21.

Δ_S(1:75) = (s_3;- 0:24) = (*medium*,-0.24). The symbolic translation α is -0.24.

2.2.3 Aggregation of 2-tuples

This process involves obtaining a single value representing a set of values of the same type; therefore, adding a series of linguistic 2-tuples must be a linguistic 2-tuple.

In the literature, we can find various 2-tuple aggregation operators (e.g., [4]) based on the classical aggregation operators, such as the arithmetic mean and weighted mean operators.

Definition 2.3. Let $x = \{(s_1, \alpha_1), \dots, (s_n, \alpha_n)\}$ be a set of 2-tuples; the extended Arithmetic Mean AM^* using the linguistic 2-tuples is computed as:

$$\begin{aligned} &AM^*\big((s_1,\alpha_1),\dots,(s_n,\alpha_n)\big),\\ &= \Delta\left(\sum_{i=1}^{n}\frac{1}{n}\Delta^{-1}(s_i,\alpha_i)\right),\\ &= \Delta\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i\right). \end{aligned} \tag{11}$$

Definition 2.4. Let $\{(s_1, \alpha_1) \dots (s_n, \alpha_n)\}$ be a set of linguistic 2-tuples, and $W = \{w_1, \dots, w_n\}$ the set of its associated weights. Then, the 2-tuple weighted mean, W_AM^*, is computed as:

$$\begin{aligned} &W_{AM^*}\big((s_1,\alpha_1),\dots,(s_n,\alpha_n)\big),\\ &= \Delta\left[\frac{\sum_{i=1}^{n}\Delta^{-1}(s_i,\alpha_i)w_i}{\sum_{i=1}^{n}w_i}\right],\\ &= \Delta\left[\frac{\sum_{i=1}^{n}\beta_i\cdot w_i}{\sum_{i=1}^{n}w_i}\right]. \end{aligned} \tag{12}$$

2.2.4 Comparison of 2-tuples

The 2-tuple information is compared using the lexicographic order. Let (s_k, α_1) and (s_l, α_2) be two 2-tuples represented by two assessments:

– If $k < I$ then (s_k, α_1) is smaller than (s_l, α_2)
– If $k = I$ then:
  1. If $\alpha_1 = \alpha_2$, then (s_k, α_1) and (s_l, α_2) represent the same value.
  2. If $\alpha_1 < \alpha_2$, then (s_k, α_1) is smaller than (s_l, α_2).
  3. If $\alpha_1 > \alpha_2$, then (s_k, α_1) is bigger than (s_l, α_2).

2.2.5 Negation Operator of a 2-tuple

The negation operator over 2-tuples is expressed as:

$$Neg(s_i, \alpha) = \Delta_S(h - \Delta_S{}^{-1}(s_i, \alpha)), \tag{13}$$

where $h + 1$ is the number of elements in $S = \{s_0, \dots, s_h\}$, $s_i \in S$.

2.3 The Linguistic Fusion Approach for Heterogeneous Information based on the 2-tuple Fuzzy Linguistic Model

The approach used in this section to fuse heterogeneous information based on the 2-tuple linguistic model considers various transformation functions that go from numerical, interval, and linguistic information sources toward a common linguistic format [21].

1 Choosing the basic linguistic term set (BLTS) $\mathrm{S_{BLTS}} = \{\mathrm{s_0}, \mathrm{s_1}, \dots, \mathrm{s_h}\}$: The BLTS must have the maximum granularity to maintain the uncertainty degree associated with the DM and the capacity of discrimination to express the preference values [9].

2 Transformation of the heterogeneous information into fuzzy sets in a linguistic domain:

Table 10. Power, weakness, and qualification scores

$ei_i Oei_j$	ei_1	ei_2	ei_3	ei_4	ei_5	ei_6	ei_7
$p_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - power	0	0	3	2	2	4	2
$f_{D}^{(s_{t_c},\alpha_c)^{(0)}}$ - weakness	5	1	0	4	1	0	2
$q_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - qualification	-5	-1	3	-2	1	4	0

Each input value x is transformed into a fuzzy set on S_{BLTS}, $F(S_{BLTS})$ by means of one of the following transformation functions: For $x \in [a,b]$, the numerical transformation function $T_{NS_{BLTS}}: [a,b] \to F(S_{BLTS})$ is expressed as:

$$T_{NS_{BLTS}}(x) = \sum_{i=0}^{h} (S_i/\lambda_i), \tag{14}$$

where $\lambda_i = \mu_{s_i}(x) \in [0,1]$ is the membership degree of x to $s_i \in S_{BLTS}$:

$$\mu_{s_i}(x) = \begin{cases} 0 & if \quad x \notin Support\ \mu_{s_i}(x) \\ \dfrac{x-a_i}{b_i-a_i} & if \quad a_i \le x \le b_{i,} \\ 1 & if \quad b_i \le x \le d_{i,} \\ \dfrac{c_i-x}{c_i-d_i} & if \quad d_i \le x \le c_i. \end{cases} \tag{15}$$

For $x \in V([a,b])$, the interval transformation function $T_{VS_{BLTS}}: V([a,b]) \to F(S_{BLTS})$ is expressed as:

$$T_{VS_{BLTS}}(x) = \sum_{i=0}^{h} (S_i/\lambda_i), \tag{16}$$

where:

$$\lambda_i = \max_y min\{\mu_V(y), \mu_{s_j}(y)\}, \tag{17}$$

$$i \in \{0,\ldots,h\}, \tag{18}$$

$$\mu_V(y) = \begin{cases} 0 & if \quad y < a, \\ 1 & if \quad a \le y \le b, \\ 0 & if \quad y > b. \end{cases} \tag{19}$$

For $x = s_j \in S$ with , $S = \{S_0, \ldots, S_h\}$ the linguistic transformation function $T_{SS_{BLTS}}: S \to F(S_{BLTS})$ is expressed as:

$$T_{SS_{BLTS}}(S_i) = \sum_{i=0}^{h} (S_i/\lambda_i), \tag{19}$$

where:

$$\lambda_i = \max_y min\{\mu_{s_j}(y), \mu_{s_i}(y)\}, \quad i \in \{0,\ldots,h\}. \tag{20}$$

This information fusion process [13] is illustrated in Figure 1.

2.4 Transformation of Fuzzy Sets into Linguistic 2-tuple Values

Here, the fuzzy sets are converted into linguistic 2-tuples over the BLTS through the function $\chi: F(S_{BLTS}) \to S \times [-0.5,0.5)$, which is stated as follows:

$$\chi(\lambda_0, \lambda_1, \ldots, \lambda_h) = \Delta_S\left(\frac{\sum_{i=0}^{h} i\lambda_i}{\sum_{i=0}^{h} \lambda_i}\right) = (s,\alpha) = \bar{s} \in \bar{S}_{BLTS}, \tag{21}$$

where $S = \{S_0, S_1, \ldots, S_h\}$ is the set of linguistic terms, and $\bar{S} = S \times [-0.5,0.5)$ is the linked 2-tuple term set. The function Δ_S is defined in section 2.2.2. After the transformations of heterogeneous information into 2-tuple linguistic values in the BLTS have been carried out, we can use the 2-tuple linguistic computation model [7] to compute linguistic results in $\bar{S}_{BLTS}$. This step uses 2-tuple linguistic aggregation operators [20, 22]. Based on the results presented in this section, we present a linguistic extension of the ELECTRE III method in the following section.

3 The Linguistic ELECTRE III Method

This section presents our proposal for the linguistic extension of the ELECTRE III model; to this end, a procedure is defined to model the partial and global concordance indices, the discordance indices by criterion, the thresholds of the criteria, and the credibility index linguistically so that they can accept 2-tuple linguistic values.

Consequently, the linguistic ELECTRE III method provides a more realistic operability of the qualitative criteria when solving multicriteria ranking problems.

Within this procedure, a linguistic difference function allows for the calculation of the linguistic difference for each pair of alternatives for each

Table 11. Crisp outranking relation for Iteration 1 in Distillation 2

ei_iOei_j	ei_1	ei_2	ei_3	ei_4	ei_5	ei_7
ei_1	0	0	0	0	0	0
ei_2	0	0	0	0	0	0
ei_3	1	0	0	1	0	1
ei_4	1	1	0	0	0	0
ei_5	1	0	0	1	0	0
ei_7	1	0	0	1	0	0

decision criterion. The linguistic output provided by the linguistic difference function serves as the linguistic input of the linguistic concordance and discordance indices for each criterion.

The output of the concordance and discordance indices are expressed in the same linguistic scale to preserve interpretability. Figure 2 schematically presents the process for modeling the linguistic outranking index in three phases.

3.1 Fusion of Heterogeneous Information

In a multicriteria ranking problem with a heterogeneous information environment, alternatives are evaluated using diverse expression domains based on the uncertainty and criteria type, as well as each DM's experience.

3.1.1 Transformation into Fuzzy Sets

The expression domains (numerical, interval, and linguistic) used in the heterogeneous framework are presented in this part of the first phase. The fusion approach handles these three types of information.

Previously, a unification domain S_{BLTS} is defined as allowing the transforming of heterogeneous information by fuzzy sets into S_{BLTS}, using the respective transformation functions of Eqs. 14, 16, and 19.

3.1.2 Transformation into 2-tuples Linguistic Values

Then, the process transforms the fuzzy sets into 2-tuple linguistic values in S_{BLTS} using Eq. (21). Hence, the fused evaluation for each criterion g_k concerning each alternative a_i, is represented in a 2-tuple linguistic value $\bar{g}_k(a_i) = (s_i, \alpha_i) \in S_{BLTS}$.

3.2 The Linguistic Difference Function D_S

This function is introduced to facilitate the computation of the linguistic concordance and discordance indices because the input values of the indices and thresholds must be linguistic values for a correct interpretation.

To compare two linguistic values, we need a comparison scale that can measure the linguistic difference between them. The scale's granularity will depend on the decision maker's knowledge, who needs to interpret the difference between the two alternatives using a bipolar scale [23]. This type of scale is convenient because it has a neutral point, which separates the positive differences from the negative ones [5].

In short, this function's linguistic output is the input of the linguistic concordance and discordance indices. The linguistic difference function is expressed in the linguistic scale, and the threshold parameters are stated accordingly. Consequently, a proper linguistic difference function between linguistic preference values is necessary for developing an extension of ELECTRE III dealing with fuzzy linguistic information.

Definition 3.1. Let $S_{BLTS} = \{s_0, \dots, s_h\}$ and $S^C = \{l_0^C, \dots, l_{\hat{h}}^C\}$ be the set of linguistic terms for preference values and the set of linguistic terms to express the linguistic difference value between two terms in S_{BLTS}, respectively. Let (s_i, α_i) and (s_j, α_j) be two 2-tuple linguistic values stated in S_{BLTS}. The linguistic difference value between (s_i, α_i) and (s_j, α_j) expressed in $\bar{S}^C$ is calculated by:

$$D_S\colon \bar{S}_{BLTS} \times \bar{S}_{BLTS} \quad \rightarrow \quad \bar{S}^C, \tag{22}$$

$$D_S\left((s_i,\alpha_i),(s_j,\alpha_j)\right) = \Delta_{S^c}\left(\frac{\left(\left(\Delta^{-1}_{\bar{S}_{BLTS}}(s_j,\alpha_j) - \Delta^{-1}_{\bar{S}_{BLTS}}(s_i,\alpha_i)\right)+h\right)}{2.h}.\hat{h}\right). \tag{23}$$

The proposed linguistic difference function satisfies the following properties:

– The difference between the same value of S_{BLTS} is the neutral point value of S^C:

$$D_s\left((s_i,\alpha_i),(s_i,\alpha_i)\right) = neutral\ point\{S^c\} = l^c_{\hat{h}/2}. \tag{24}$$

– The difference between the minimum (maximum) and maximum (minimum) values of S_{BLTS} must be the maximum (minimum) value of S^C:

$$D_s((s_0,0),(s_h,0)) = max\{S^c\} = l^c_{\hat{h}},\\ D_s((s_h,0),(s_0,0)) = min\{S^c\} = l^c_0. \quad (25)$$

The proof of these properties is trivial. We propose the following syntax for S^C:

$$S^c = \{l^c_0: Extremely\_Lower(EL),\\ l^c_1: Much_Lower(ML),\\ l^c_2: Lower(L), l^c_3: Slightly\_Lower(SL),\\ l^c_4: Identical(I), l^c_5: Slightly_{Higher(SH)},\\ l^c_6: Higher(H),\\ l^c_7: Much\_Higher(MH),\\ l^c_8: Extremely\_Higher(EH)\}. \quad (26)$$

Example 2. In this example, we perform the linguistic difference of the 2-tuple linguistic values $(L,-0.2)$ and $(H,-0{,}3)$ using the linguistic difference function (Definition 3.1). The set of linguistic terms for preference values is defined as follows:

$$S_{BLTS}\\ = \{s_0: Very\_low(VL),\ s_1: Low(L),\ s_2: Medium(M),\\ s_3: High(H),\ s_4 Very\_high(VH)\}. \quad (27)$$

With the linguistic terms S^C to describe the difference function defined above:

$$D_s((L,-0.2),(H,-0.3)) = \Delta_{s^c}\left(\frac{((2.7-0.8)+4)}{(2).(4)}.8\right)\\ = \Delta_{s^c}(5.9) = (l^c_5, 0.9) = (SH, 0.9). \quad (28)$$

3.3 Linguistic Concordance and Discordance Indices

The linguistic concordance and discordance indices defined in this section have 2-tuple linguistic values as input and output.

3.3.1 The Linguistic Concordance Index

The linguistic concordance value of $(a_i,a_j) \in A \times A$ for a criterion k is calculated by making use of a linguistic concordance index $\bar{C}_k(a_i,a_j)$ through the linguistic difference function. The input value of the $k-th$ concordance index is a linguistic difference value of:

Table 12. Power, weakness, and qualification scores for Iteration 1 in Distillation 2

$ei_i O ei_j$	ei_1	ei_2	ei_3	ei_4	ei_5	ei_7
$p_{D_0}^{(s_c,\alpha_c)^{(0)}}$ - power	0	0	3	2	2	2
$f_D^{(s_c,\alpha_c)^{(0)}}$ - weakness	4	1	0	3	0	2
$q_{D_0}^{(s_c,\alpha_c)^{(0)}}$ - qualification	-4	-1	3	-1	2	0

$$(a_i,a_j) \in A \times A\\ D_s\left(\bar{g}_k(a_i), \bar{g}_k(a_j)\right) \in \bar{S}^c, \quad (29)$$

where $g_k: A \to S_{BLTS}$ is the $k-th$ criteria function. The output value of the linguistic concordance function, i.e., the concordance value, is likewise represented by a value in a linguistic concordance scale $\bar{S}^P$.

$S^P = \{s^P_0, \ldots, s^P_{h_P}\}$ represents a linguistic term set. The granularity of S^P is chosen following the DM's knowledge to make clear the concordance value of $(a_i,a_j) \in A \times A$.

Here, $(s^P_0, 0)$ represents no concordance and $(s^P_{h_P}, 0)$ strict concordance.

The concordance index $\bar{C}_k(a_i,a_j)$ varies from $(s^P_0,0)$ to $(s^P_{h_P},0)$. If $\bar{C}_k(a_i,a_j)$ is equal to $(s^P_0,0)$, then a_i is worse than a_j.

The indifference and preference thresholds $\left(l_{t_{q_k}}, \alpha_{q_k}\right)_q$ and $\left(l_{t_{p_k}}, \alpha_{p_k}\right)_p$ respectively, both in $\bar{S}^C$, are used to construct a concordance index $\bar{C}_k(a_i,a_j)$ for each criterion k , defined by:

Definition 3.2. *The linguistic concordance index concerning a criterion* g_k, $\bar{C}_k(a_i,a_j)$, *that symbolizes the linguistic concordance value stated in 2-tuple linguistic values in* $S^P = \{s^P_0, \ldots, s^P_{h_P}\}$ *of the linguistic difference value between* a_j *over* a_i, *regarding criterion* k, $\bar{D}_s(a_i,a_j)_k = D_s\left(\bar{g}_k(a_i), \bar{g}_k(a_j)\right) \in \bar{S}^c$ *is specified as:*

$$\bar{C}_k(a_i,a_j): A \times A \to \bar{S}^P, \quad (30)$$

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Table 13. Crisp outranking relation for Iteration 1 in Distillation 3

ei_iOei_j	ei_1	ei_2	ei_4	ei_5	ei_7
ei_1	0	0	0	0	0
ei_2	0	0	0	0	0
ei_4	1	1	0	0	0
ei_5	1	0	1	0	0
ei_7	1	0	1	0	0

$$\bar{C}_k(a_i, a_j) = \begin{cases} (S_0^P, 0), & if \quad \bar{D}_s(a_i, a_j)_k > (l_{t_{p_k}}, \alpha_{p_k})_p \\ \Delta_{S^P}\Big(\dfrac{\Delta_{S^C}^{-1}(l_{t_{p_k}}, \alpha_{p_k})_p - \frac{\hat{h}}{2} - [\Delta_{S^C}^{-1}(\bar{D}_s(a_i, a_j}{\Delta_{S^C}^{-1}(l_{t_{p_k}}, \alpha_{p_k})_p - \frac{\hat{h}}{2}(\ \Delta_{S^C}^{-1}(l_{t_{q_k}}, \alpha_{q_k}} & \\ \quad if \ (l_{t_{q_k}}, \alpha_{q_k})_q < \bar{D}_s(a_i, a_j)_k \le (l_{t_p} & \\ (S_{h_p}^P, 0), & if \quad \bar{D}_s(a_i, a_j)_k \le (l_{t_{q_k}}, \alpha_{q_k} \end{cases} \tag{31}$$

With $k = 1, \dots, n$.

The concordance index $\bar{C}_k(a_i, a_j)$ is a linguistic index measuring whether "a_i is at least as good as a_j" on criterion k.

3.3.2 The Linguistic Discordance Index

For each criterion g_k a linguistic discordance index $\bar{d}_k(a_i, a_j)$ can be defined. This index measures how much g_k is more or less discordant with the affirmation "a_i outranks a_j." This index considers a linguistic veto threshold $(l_{t_{v_k}}, \alpha_{v_k})_v$ to calculate linguistic concordance.

It should be mentioned that any outranking of a_i by a_j by specified by the concordance index can be overruled if there is any criterion g_k for which the alternative a_j outperforms the alternative a_i by at least a veto threshold, even if all the other criteria favor the outranking of a_i ($\bar{D}_s(a_i, a_j)_k \ge (l_{t_{v_k}}, \alpha_{v_k})$).

So, if a_i is better than a_j normally, there may be some criteria (possibly one) where a_i is worse than a_j. The index $\bar{d}_k(a_i, a_j)$ displays this condition for that criterion. $\bar{d}_k(a_i, a_j)$ varies from $(s_0^P, 0)$ to $(s_{h_P}^P, 0)$. $(s_0^P, 0)$ represents no discordance and $(s_{h_P}^P, 0)$ represents strict discordance. The linguistic discordance index is calculated according to the following definition:

Definition 3.3. *The linguistic discordance index $\bar{d}_k(a_i, a_j)$, for a criterion g_k, that represents the linguistic discordance value expressed in 2-tuple linguistic values in $S^P = \{s_0^P, \dots, s_{h_P}^P\}$ of the linguistic difference value between a_j over a_i, regarding criterion k, $\bar{D}_S(a_i, a_j)_k = D_S(\bar{g}_k(a_i), \bar{g}_k(a_j)) \in \bar{S}^C$ is stated as:*

$$\bar{d}_k(a_i, a_j): A \times A \to \bar{S}^P, \tag{32}$$

$$\bar{d}_k(a_i, a_j) = \begin{cases} (S_0^P, 0), & if \quad \bar{D}_s(a_i, a_j)_k \le (l_{t_{p_k}}, \alpha_{p_k})_p \\ \Delta_{S^P}\left(\dfrac{\Delta_{S^C}^{-1}\left(\bar{D}_s(a_i, a_j)_k\right) - \frac{\hat{h}}{2} - \left[\Delta_{S^C}^{-1}(l_{t_{p_k}}, \alpha_{p_k})_p - \frac{\hat{h}}{2}\right]}{\Delta_{S^C}^{-1}(l_{t_{v_k}}, \alpha_{v_k})_v - \frac{\hat{h}}{2} - \left(\Delta_{S^C}^{-1}(l_{t_{p_k}}, \alpha_{p_k})_p - \frac{\hat{h}}{2}\right)} . h_p \right), & \\ \quad if \ (l_{t_{p_k}}, \alpha_{p_k})_p < \bar{D}_s(a_i, a_j)_k < (l_{t_{v_k}}, \alpha_{v_k})_v & \\ (S_{h_p}^P, 0), & if \quad \bar{D}_s(a_i, a_j)_k \ge (l_{t_{v_k}}, \alpha_{v_k})_v \end{cases} \tag{33}$$

with $k = 1, \dots, n$.

3.4 The Linguistic Outranking Relation in the Linguistic ELECTRE III

The linguistic outranking relation $O_A^{\bar{\sigma}}$ defined for each $(a_i, a_j) \in A \times A$ as a linguistic credibility index, $\bar{\sigma}(a_i, a_j)$, state broadly in what linguistic measure "a_i outranks a_j" employing both the linguistic concordance index $\bar{C}(a_i, a_j)$ and the linguistic discordance indices $\bar{d}_k(a_i, a_j)$ for each criterion g_k.

The linguistic credibility index is the comprehensive linguistic concordance index reduced by the linguistic discordance indices. In the nonappearance of such linguistic discordance criteria, $\bar{\sigma}(a_i, a_j) = \bar{C}(a_i, a_j)$.

This linguistic credibility value is decreased in the occurrence of one or more linguistic discordant criteria g_k when $\bar{d}_k(a_i, a_j) > \bar{C}(a_i, a_j)$. In correspondence with the veto effect $\bar{\sigma}(a_i, a_j) = (s_0^P, 0)$ if exists a linguistic discordance index such that $\bar{d}_k(a_i, a_j) = (s_{h_P}^P, 0)$, does not matter what the weight of the criterion w_k is. The linguistic credibility index $\bar{\sigma}(a_i, a_j)$ is defined as follows:

Table 14. Power, weakness, and qualification scores for Iteration 1 in Distillation 3

$ei_i O ei_j$	ei_1	ei_2	ei_4	ei_5	ei_7
$p_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - power	0	0	2	2	2
$f_{D}^{(s_{t_c},\alpha_c)^{(0)}}$ - weakness	3	1	2	0	0
$q_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - qualification	-3	-1	0	2	2

$$\bar{\sigma}(a_i,a_j) = \begin{cases} \bar{C}(a_i,a_j), if\ \bar{K}(a_i,a_j)=\phi \\ \Delta_{S^P}(\Delta_{S^P}^{-1}(\bar{C}(a_i,a_j)) \bullet \\ \prod_{k\in\bar{K}(a_i,a_j)} \frac{\Delta_{S_P}^{-1}(s_{h_P}^P,0)-\Delta_{S_P}^{-1}\left(\bar{d}_k(a_i,a_j)\right)}{\Delta_{S_P}^{-1}(s_{h_P}^P,0)-\Delta_{S_P}^{-1}\left(\bar{C}(a_i,a_j)\right)}) \\ if\ \bar{K}(a_i,a_j)\neq\phi, \end{cases} \quad (34)$$

where:

$$\bar{K}(a_i,a_j)=\{g_k\in G|\bar{d}_k(a_i,a_j)>\bar{C}(a_i,a_j)\}. \quad (35)$$

The formula for determining the linguistic value of $\bar{\sigma}(a_i,a_j)$ over the linguistic interval $((s_0^P,0),(s_{h_P}^P,0))$ is non-compensatory, i.e., an alternative's notable poor performances in some criteria cannot be compensated for even with very high performance in other criteria. The aggregated performance exposes this fact. This completes the first phase of the linguistic ELECTRE III method.

3.5 The Ranking Algorithm in the Linguistic Extension of ELECTRE III

The second phase of the linguistic ELECTRE method is to exploit the linguistic outranking relation $O_A^{\bar{\sigma}}$ to get a partial preorder of the alternatives. This final partial preorder is obtained because of the "intersection" of two complete preorders resulting from the descending and ascending distillations [24].

In the descending distillation, the procedure ranks the alternatives from the best to the worst; on the contrary, in the ascending distillation, the process ranks the alternatives from the worst to the best.

In the following, we modify the distillation procedure of ELECTRE III. In the linguistic ELECTRE III distillation procedure, we state a set of linguistic credibility cutting levels $(s_{t_c},\alpha_c)^{(r)}$ in $S^P=\{s_0^P,\dots,s_{h_P}^P\}$. Given a *linguistic cutoff* level symbolized by $(s_{t_c},\alpha_c)^{(r)}$, both distillations relate to the following linguistic crisp outranking relation:

$$a_i O_A^{(s_{t_c},\alpha_c)^{(r)}} a_j \quad (36)$$

$$\Leftrightarrow \begin{cases} \bar{\sigma}(a_i,a_j)\geq(s_{t_c},\alpha_c)^{(r)} \\ \bar{\sigma}(a_i,a_j)>\Delta_{S^P}\left(\Delta_{S^P}^{-1}(\bar{\sigma}(a_j,a_i))+z(\bar{\sigma}(a_i,a_j))\right), \end{cases}$$

where $z(\bar{\lambda})=\frac{\alpha(\Delta_{S^P}^{-1}(\bar{\lambda}))}{h_P}+\beta$ is a linguistic distillation threshold and α and β are two distillation coefficients. $(s_{t_c},\alpha_c)^{(r)}$ is also a linguistic preference parameter, which fixes the minimum degree of credibility considered obligatory by the DM to support the statement "a_i outranks a_j". From the linguistic crisp outranking relation, for each alternative a_i, its $(s_{t_c},\alpha_c)^{(r)}$-qualification is:

$$q_A^{(s_{t_c},\alpha_c)^{(r)}}(a_i)=p_A^{(s_{t_c},\alpha_c)^{(r)}}(a_i) - f_A^{(s_{t_c},\alpha_c)^{(r)}}(a_i), \quad (37)$$

where $p_A^{(s_{t_c},\alpha_c)^{(r)}}(a_i)=\left|\left\{a_j\in A: a_i O_A^{(s_{t_c},\alpha_c)^{(r)}} a_j\right\}\right|$ is the $(s_{t_c},\alpha_c)^{(r)}$-power of a_i; it is the number of alternatives that are outranked by a_i, and $f_A^{(s_{t_c},\alpha_c)^{(r)}}(a_i)=\left|\{a_j\in A: a_j O_A^{(s_{t_c},\alpha_c)^{(r)}} a_i\}\right|$ is the $(s_{t_c},\alpha_c)^{(r)}$-weakness of a_i; it is the number of alternatives outranking a_i.

In the rest of this section, we explain the descending and ascending distillation algorithms in detail as follows: Let $(s_{t_c},\alpha_c)^{(1)}$ be the first fixed *linguistic cutoff* level and $q_A^{(s_{t_c},\alpha_c)^{(1)}}(a_i)$ be the qualification of alternative a_i. Then, choose in A the best ones resulting a subset of alternatives from A that has the maximum qualification (descending selection, $\overrightarrow{D}_1$) or the worst alternatives resulting thus a subset of alternatives from A, which has the minimum qualification (ascending selection, $\overleftarrow{D}_1$):

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Table 15. Crisp outranking relation for Iteration 2 in distillation 3

$ei_i O ei_j$	ei_5	ei_7
ei_5	0	1
ei_7	0	0

$$\vec{D}_1 = \left\{ a_i \in A | q_A^{(s_{t_c},\alpha_c)^{(1)}}(a_i) = \vec{q}_A = \max_{x \in A} q_A^{(s_{t_c},\alpha_c)^{(1)}}(x) \right\}, \quad (38)$$

$$\overleftarrow{D}_1 = \left\{ a_i \in A \left| q_A^{(s_{t_c},\alpha_c)^{(1)}}(a_i) = \overleftarrow{q}_A = \min_{x \in A} q_A^{(s_{t_c},\alpha_c)^{(1)}}(x) \right. \right\}. \quad (39)$$

Consequently, at the end of the k steps of the first distillation, the first subset of A is obtained, representing the first (last) class of one of the two final preorders. Let $\vec{C}_1 = \vec{D}_k$ symbolize the first class of the descending selection, and $\overleftarrow{C}_1 = \overleftarrow{D}_k$ indicate the last class of the ascending selection.

Let $\vec{A}_1 = A \backslash \vec{C}_1$, or $\overleftarrow{A}_1 = A \backslash \overleftarrow{C}_1$ represent the remaining subset of the alternatives from *A* to rank after the first distillation. In $\vec{A}_1$ and $\overleftarrow{A}_1$ The alternatives' qualification is computed again for choosing one or various alternatives. This process is reiterated until all the alternatives are ranked.

The distillation process is condensed in the following way:

1. Set $n = 0$, put or $\vec{A}_0 = A$ (descending), or $\overleftarrow{A}_0 = A$ (ascending).
2. Set: $(s_{t_c}, \alpha_c)^{(0)} = \max\limits_{\substack{a_i, a_j \in \vec{A}_n \\ a_i \neq a_j}} \bar{\sigma}(a_i, a_j) \quad or \quad (s_{t_c}, \alpha_c)^{(0)} = \max\limits_{\substack{a_i, a_j \in \overleftarrow{A}_n \\ a_i \neq a_j}} \bar{\sigma}(a_i, a_j)$.
3. Put $k = 0,\ D_0 = \vec{A}_n$ (descending) or $D_0 = \overleftarrow{A}_n$ (ascending).
4. Choose the maximum value from the linguistic credibility scores that are less than $(s_{t_c}, \alpha_c)^{(k)} - z((s_{t_c}, \alpha_c)^{(k)})$.
5. $(s_{t_c}, \alpha_c)^{(k+1)} = \max\limits_{\substack{\{\bar{\sigma}(a_i,a_j) + z((s_{t_c},\alpha_c)^{(k)}) < (s_{t_c},\alpha_c)^{(k)}\} \\ a_i, a_j \in D_k}} \bar{\sigma}(a_i, a_j)$.
6. $If\ \forall a_i, a_j \in D_k,\ \bar{\sigma}(a_i, a_j) + z((s_{t_c}, \alpha_c)^{(k)}) > (s_{t_c}, \alpha_c)^{(k)}, \quad put \quad (s_{t_c}, \alpha_c)^{(k+1)} = (s_0, 0)$.
7. Calculate the $(s_{t_c}, \alpha_c)^{(k+1)}$-qualifications $(q_A^{(s_{t_c},\alpha_c)^{(k)}}(a_i)) \forall a_i \in D_k$.
8. Obtain the maximum or minimum $(s_{t_c}, \alpha_c)^{(k+1)}$-qualification score: $\vec{q}_{D_k} = \max\limits_{x \in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)$ (descending) or $\overleftarrow{q}_{D_k} = \min\limits_{x \in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)$ (ascending).
9. Construct $\vec{D}_{k+1} = \left\{ a_i \in D_k \left| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a_i) = \vec{q}_{D_k} \right. \right\}$ (descending) or $\overleftarrow{D}_{k+1} = \left\{ a_i \in D_k \left| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a_i) = \overleftarrow{q}_{D_k} \right. \right\}$ (ascending).
10. If $\left|\vec{D}_{k+1}\right| = 1 \ \ or \ \ \left|\overleftarrow{D}_{k+1}\right| = 1 \ \ or \ \ (s_{t_c}, \alpha_c)^{(k+1)} = (s_0, 0)$ you proceed to step (9).
11. else, do $k = k + 1, \quad D_k = \vec{D}_k$ (descending) or $D_k = \overleftarrow{D}_k$ (ascending) and go to step (4).
12. $\vec{C}_{n+1} = \vec{D}_{k+1}$is the set of alternatives carried through the $(n+1) - th$ downward distillation, termed the $(n+1) - th$ distillate of the downward procedure. $\overleftarrow{C}_{n+1} = \overleftarrow{D}_{k+1}$ is the set of alternatives taken through the $(n+1) - th$ upward distillation, termed the $(n+1) - th$ distillate of the upward procedure.
13. Put $\vec{A}_{n+1} = \vec{A}_n \backslash \vec{C}_{n+1}$ (descending) or $\overleftarrow{A}_{n+1} = \overleftarrow{A}_n \backslash \overleftarrow{C}_{n+1}$ (ascending).
14. If $\vec{A}_{n+1} \neq \phi, \quad or \quad \overleftarrow{A}_{n+1} \neq \phi$ then $n = n + 1$, and proceed to Step (2).
15. Otherwise, end of the distillation.

During the same distillations, when advancing from step k to step $k + 1$, the linguistic cutoff level $(s_{t_c}, \alpha_c)^{(k)}$ is replaced by $(s_{t_c}, \alpha_c)^{(k+1)} < (s_{t_c}, \alpha_c)^{(k)}$ as follows (D_k is the remaining set of alternatives to rank):

$$(s_{t_c}, \alpha_c)^{(k+1)} = \max_{\substack{\{\bar{\sigma}(a_i,a_j) + z((s_{t_c},\alpha_c)^{(k)}) < (s_{t_c},\alpha_c)^{(k)}\} \\ a_i, a_j \in D_k,}} \bar{\sigma}(a_i, a_j) \quad (40)$$

where $z(\,(s_{t_c}, \alpha_c)^{(k)}) = \Delta_{S^P}\left(\frac{\alpha(\Delta_{S^P}^{-1}((s_{t_c},\alpha_c)^{(k)}))}{h_P} + \beta\right)$.

The analyst can fix one value for the distillation coefficients α and β before the computations. The standard values recommended in the literature are $\alpha = -0.15$ and $\beta = 0.30$.

We obtain two complete preorders at the end of the distillation procedure. In each preorder, the alternatives are regrouped in a partition of

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Table 16. Power, weakness, and qualification scores for Iteration 2 in distillation 3

$ei_i O ei_j$	ei_5	ei_7
$p_{D_0}^{(s_c,\alpha_c)^{(0)}}$ **- power**	1	0
$f_D^{(s_c,\alpha_c)^{(0)}}$ **- weakness**	0	1
$q_{D_0}^{(s_c,\alpha_c)^{(0)}}$ **- qualification**	1	-1

equivalence classes, forming a ranking from the best to the worst alternatives.

Each class includes at least one alternative. A partial preorder of the alternatives is constructed utilizing the intersection of both preorders, which specifies the comparisons between alternatives and emphasizes the possible incomparabilities as follows:

- Alternative a_i is preferred to the alternative a_j if a_i belongs to a class not worse than alternative a_j in both preorders and a better class for at least one of the two preorders.
- Alternative a_i is indifferent to alternative a_j if a_i and a_j belong to the same class in the two preorders.
- Alternatives a_i and a_j are incomparable if a_i belongs to a class better than a_j in one preorder and worse in the other or vice *versa*.

To illustrate the proposed method, we present in the following section a step-by-step example of the linguistic ELECTRE III method for ranking a set of alternatives.

4 An Illustrative Example

We will use a case study from [25] to demonstrate the proposed approach. This case study is an Environmental Impact Significance Assessment problem in which heterogenous data (qualitative and quantitative judgments) obtained from a DM are used to determine the environmental impacts that a set of projects or industrial activities can have on a petrol station's usual operations.

This case study aims to evaluate seven ecological effects that can occur between the interactions of four industrial activities and four environmental factors in a petrol station. The evaluation seeks to rank the identified impacts from the most to the least significant. Each step of the linguistic extension of the ELECTRE III method is explained below.

Step 1. Formulation of the multicriteria ranking problem. Given a set of activities from a petrol station $A = \{a_1$: The operation of petrol pumps, a_2: the operation of the car wash, a_3: the transport of fuel and materials, and a_4: the filling of fuel tanks$\}$ and four possible environmental factors $F = \{f_1$: Daily sound comfort, f_2: hydrocarbons in the air, f_3: public health and civic safety, and f_4: energy infrastructures$\}$ a set of seven possible environmental impacts that are triggered from the interaction between A and F, was identified $EI = \{(a_1, f_2), (a_1, f_3), (a_2, f_1), (a_2, f_4), (a_3, f_1), (a_3, f_4), (a_4, f_1)\}$.

For convenience, we define $EI = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_6, ei_7\}$. For the assessment of the elements in *EI*, a DM, which has specific knowledge is public health, expressed his preferences on *EI* using diverse expression domains: Numerical(N), Interval-valued(I), or Linguistic (L) over a set of 10 criteria defined in Table 1. Note that the preference direction for all criteria is to maximize.

The DM uses a linguistic domain with five linguistic terms denoted by S^5 to express his/her preferences. Each linguistic term set is symmetrically and uniformly distributed, and its syntax is defined in the following form:

$$S^5 = \begin{cases} S_0 : VeryLow(VL),\ S_1 : Low(L),\ S_2 : Mediu \\ S_3 : High(H), S_4 : VeryHigh(VH). \end{cases} \quad (41)$$

Step 2. Collecting the heterogeneous information: The DM assessed each criterion for each impact in EI using a heterogeneous framework. The expression domain used for each criterion was according to its nature; for criteria g_1, g_2, g_6, g_7, g_8, g_9, and g_{10} were used the linguistic terms in S^5; for criteria g_3 and g_5 were used a scale based on real numbers; meanwhile, for criterion g_4 was used an interval scale. The assessment made by the DM is presented in Table 2.

Step 3. Fusion of the heterogeneous information: The chosen linguistic domain to fuse the information is S^5. The integrated information given by the DM is shown in Table 3.

Step 4. Computing linguistic difference values between unified assessments: The linguistic difference value between a pair of 2-tuple linguistic values is stated in the linguistic comparison scale S^C presented in Figure 3. Linguistic difference values are calculated by Eq. (23).

Step 5. Computing linguistic concordance values: Linguistic concordance index concerning a criterion g_k, $C_k(ei_i, ei_j)$.

Calculations to get individual linguistic concordance values. Initially, the linguistic preference scale S^P is chosen. After that, for each criterion g_k, its linguistic concordance function is performed (Eq. (31)) and its indifference and preference threshold parameters are defined in 2-tuple linguistic values in S^C.

Each linguistic concordance value for each alternative ei_i, with regards to alternative ei_j, over a criterion g_k, is calculated using the linguistic outranking function (Eq. 31).

The calculation of the linguistic outranking value for each criterion is given in a linguistic preference scale $S^P = \{S_0^P, \dots, S_k^P\}$ with nine linguistic terms. The inter-criteria parameters of $g_k, k = 1,2,\dots,10$ are presented in Table 4, which are described in S^C.

Example 3. Calculation of $\bar{C}_1(ei_1, ei_6)$. According to the descriptive example, the computations of the linguistic concordance indices (Definition 3.2) can be made in the following form: From Table 3, $\bar{g}_1(ei_1) = (L, 0)$ and $\bar{g}_1(ei_6) = (VH, 0)$, then:

$$\begin{aligned} D_S(ei_1, ei_6)_1 &= D_S\left(\bar{g}_1(ei_1), \bar{g}_1(ei_6)\right)_1 \\ &= D_S\left((L,0),(VH,0)\right)_1 \\ &= \Delta_{S^C}\left(\frac{\left(\Delta^{-1}_{S_{BLTS}}(VH,0) - \Delta^{-1}_{S_{BLTS}}(L,0)\right) + 8}{2x8} \times 8\right) \\ &= \Delta_{S^C}\left(\frac{(4-1)+8}{2x8} \times 8\right) \\ &= \Delta_{S^C}(5.5) \\ &= (I_5^C, 0.5). \end{aligned} \tag{42}$$

Based on Eq. (31), since:

$$(I_5^C, 0)q \leq D_S(ei_1, ei_6)_1 = (I_5^C, 0.5) \geq (I_6^C, 0)p. \tag{43}$$

Then from interpolation, we calculate:

$$\bar{C}_1(ei_1, ei_6) = \left(\frac{(I_6^C,0)p - \frac{8}{2} - \left((I_5^C, 0.5) - \frac{8}{2}\right)}{(I_6^C,0)p - \frac{8}{2} - \left((I_5^C, 0)q - \frac{8}{2}\right)}\right) \times 8 = (I_4^C, 0). \tag{44}$$

In this way, it is possible to get the linguistic concordance indices $\bar{C}_k(ei_i, ei_j)$ on a criterion g_k for all pairs of alternatives (ei_i, ei_j), and, finally, display the linguistic concordance matrices for each criterion.

For example, on the criterion g_1 the linguistic concordance matrix is expressed in Table 5. *The comprehensive linguistic concordance index* $\bar{C}(ei_i, ei_j)$.

The comprehensive linguistic concordance index $\bar{C}(ei_i, ei_j)$ is computed using the weight vector:

$$W = (0.36, 0.24, 0.08, 0.04, 0.04, 0.04, 0.04, 0.04, 0.08, 0.04)$$

for the family of criteria. The value of $\bar{C}(ei_1, ei_6)$ is computed as follows:

$$\begin{aligned} \bar{C}(ei_1, ei_6) = \Delta_{S^P}\Big(& 0.36(\Delta^{-1}_{S^P}(\bar{C}_1(ei_1, ei_6)) \\ & + 0.24(\Delta^{-1}_{S^P}(\bar{C}_2(ei_1, ei_6)) \\ & + 0.08(\Delta^{-1}_{S^P}(\bar{C}_3(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_4(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_5(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_6(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_7(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_8(ei_1, ei_6)) \\ & + 0.08(\Delta^{-1}_{S^P}(\bar{C}_9(ei_1, ei_6)) \\ & + 0.04(\Delta^{-1}_{S^P}(\bar{C}_{10}(ei_1, ei_6))\Big) \\ = {} & (S_5^P, 0.8048). \end{aligned} \tag{45}$$

Proceeding in the same way, for all pairs of environmental impacts (ei_i, ei_j) representing the illustrative example, the comprehensive linguistic concordance matrix is obtained (Table 6).

Step 6. Computing linguistic discordance values $\bar{\mathrm{d}}_{\mathrm{k}}(\mathrm{ei_i}, \mathrm{ei_j})$. The linguistic veto thresholds have been defined on criteria g_1 and g_2. These criteria can give a linguistic discordance index that is not null.

Example 4. Calculation of $\bar{d}_1(ei_1, ei_6)$. The computations of the linguistic discordance indices (Definition3.3) can be made as follows: from Table 3 $\bar{g}_1(ei_1) = (L, 0)$ and $\bar{g}_1(ei_6) = (VH, 0)$, from example 2 $D_S(ei_1, ei_6)_1 = (I_5^C, 0.5)$.

Table 17. Crisp outranking relation for iteration 1 in distillation 4

ei_iOei_j	ei_1	ei_2	ei_4	ei_7
ei_1	0	0	0	0
ei_2	0	0	0	0
ei_4	1	1	0	0
ei_7	1	0	1	0

Table 18. Power, weakness, and qualification scores for Iteration 1 in distillation 4

ei_iOei_j	ei_1	ei_2	ei_4	ei_7
$p_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - power	0	0	2	2
$f_D^{(s_{t_c},\alpha_c)^{(0)}}$ - weakness	2	1	1	0
$q_{D_0}^{(s_{t_c},\alpha_c)^{(0)}}$ - qualification	-2	-1	1	2

Since $D_S(ei_1, ei_6)_1 = (I_5^C, 0.5) \leq (I_6^C, 0)_P$ then from Eq. 42 $\bar{d}_1(ei_1, ei_6) = (S_0^P, 0)$.

In the same way, with this computation process, it is possible to obtain the linguistic discordance indices $\bar{d}_k(ei_i, ei_j)$ on the criterion g_k for all pairs of environmental impacts (ei_i, ei_j) and display the linguistic discordance matrices for each criterion where it is defined a linguistic veto threshold. For instance, on the criterion g_1, the computed discordance matrix is defined in Table 7.

Step 7. Computing the linguistic outranking relation. Based on the comprehensive linguistic concordance matrix and the partial linguistic discordance matrices, the value of $\bar{\sigma}(ei_1, ei_6)$ is computed as follows:

Since $\bar{K}(ei_1, ei_6) = 0$ and $\forall k, \bar{d}_k(ei_1, ei_6) < \bar{C}(ei_1, ei_6)$, then, from Eq. (21), $\bar{\sigma}(ei_1, ei_6) = \bar{C}(ei_1, ei_6) = (S_5^P, 0.8048)$.

For all pairs of alternatives representing the illustrative example, the linguistic credibility matrix or linguistic outranking matrix is obtained (see Table 8).

Step 8. Ranking of alternatives from the linguistic outranking relation $\mathrm{O}_{\mathrm{A}}^{\sigma}$. The ranking algorithm can be applied according to the linguistic credibility matrix obtained by the linguistic ELECTRE III (Table 6).

For illustration purposes, we describe the procedure followed to perform the first four descending distillations as follows:

Let:

$$\overrightarrow{EI_0} = EI = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_6, ei_7\}, z(\bar{\lambda}) = \Delta_{S^p}\left(\frac{\alpha\left(\Delta_{S^p}^{-1}(\bar{\lambda})\right)}{h_p} + \beta\right). \tag{46}$$

With $\alpha = -0.15, \beta = 0.30$.

Distillation 1.

Step 1: Let $n = 0, \overrightarrow{EI_0} = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_6, ei_7\}$, $k = 0$, then $(S_{t_c}, \alpha_c)^{(0)} = \max_{\substack{a,b \in \overrightarrow{EI_0} \\ a \neq b}} \bar{\sigma}(a, b) = (S_8^P, 0)$, and $D_0 = \overrightarrow{EI_0}$, hence $(S_{t_c}, \alpha_c)^{(1)} = \max_{\substack{\{\sigma(a,b) + z((S_{t_c},\alpha_c)^{(0)}) < (S_{t_c},\alpha_c)^{(0)}\} \\ a,b \in D_0}} \bar{\sigma}(a, b) = (S_7^P, 0.84)$. Given this linguistic cutoff, we can create a linguistic crisp outranking relation using Eq. (36). Table 9 shows the resulting crisp outranking relation. From the crisp outranking relation, we calculate the $(S_{t_c}, \alpha_c)^{(1)}$-qualifications $\left(q_A^{(S_{t_c},\alpha_c)^{(0)}}(a)\right) \forall a \in D_0$ Table 10 shows the calculated power, weakness, and qualification.

The maximum $(S_{t_c}, \alpha_c)^{(k+1)}$-qualification score is $\vec{q}_{D_k} = \max_{x \in D_k} q_{D_k}^{(S_{t_c},\alpha_c)^{(k)}}(x) = 4$, then: $\vec{D}_{k+1} = \left\{a \in D_k \middle| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a) = \vec{q}_{D_k}\right\}$, $\vec{D}_1 = \{ei_6\}$. Because $|\vec{D}_1| = 1$ then $\vec{C}_1 = \vec{D}_1 = \{ei\}$ and the first distillation is completed. For the subsequent distillation, put $\overrightarrow{EI_1} = \overrightarrow{EI_0} \backslash \vec{C}_1 = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_6, ei_7\} \backslash \{ei_6\} = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_7\}$ and do do $n = n + 1 = 1$.

Distillation 2.

Step 1: Let $k = 0, D_0 = \overrightarrow{EI_1} = \{ei_1, ei_2, ei_3, ei_4, ei_5, ei_7\}$, then $(S_{t_c}, \alpha_c)^{(0)} = \max_{\substack{a,b \in \overrightarrow{EI_1} \\ a \neq b}} \bar{\sigma}(a, b) = (S_8^P, 0)$, and $(S_{t_c}, \alpha_c)^{(1)} = \max_{\substack{\{\sigma(a,b) + z((S_{t_c},\alpha_c)^{(0)}) < (S_{t_c},\alpha_c)^{(0)}\} \\ a,b \in D_0}} \bar{\sigma}(a, b) = (S_7^P, 0.84)$.

From this linguistic cutoff, we create the linguistic crisp outranking relation shown in Table 11. Then we calculate the $(S_{t_c}, \alpha_c)^{(1)}$-qualifications $\left(q_A^{(S_{t_c},\alpha_c)^{(0)}}(a)\right) \forall a \in D_0$. Table 12 shows the calculated power, weakness, and qualification for D_0 in step one of distillation 2. The maximum $(s_{t_c},\alpha_c)^{(k+1)}$-qualification score is $\vec{q}_{D_k}=\max_{x\in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)=3$, then $\vec{D}_{k+1}=\left\{a\in D_k \middle| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a)=\vec{q}_{D_k}\right\}$, $\vec{D}_1=\{ei_3\}$.

Because $|\vec{D}_1|=1$ then $\vec{C}_2=\vec{D}_1=\{ei_3\}$ and the first distillation is completed. For the subsequent distillation, put $\overrightarrow{EI}_2=\overrightarrow{EI}_1\backslash\vec{C}_2=\{ei_1,ei_2,ei_3,ei_4,ei_5,ei_7\}\backslash\{ei_3\}=\{ei_1,ei_2,ei_4,ei_5,ei_7\}$ and do do $n=n+1=2$.

Distillation 3.

Iteration 1: Let $k=0, D_0=\overrightarrow{EI_2}=\{ei_1,ei_2,ei_4,ei_5,ei_7\}$, then $(s_{t_c},\alpha_c)^{(0)}=\max_{\substack{a,b\in\overrightarrow{EI_2}\\a\neq b}}\bar{\sigma}(a,b)=(S_8^P,0)$, and $(s_{t_c},\alpha_c)^{(1)}=\max_{\substack{\{\sigma(a,b)+z((s_{t_c},\alpha_c)^{(0)})<(s_{t_c},\alpha_c)^{(0)}\}\\a,b\in D_0}}\bar{\sigma}(a,b)=(S_7^P,0.84)$.

From this linguistic cutoff, we create the linguistic crisp outranking relation shown in Table 13. Then we calculate the $(s_{t_c},\alpha_c)^{(1)}$-qualifications $\left(q_A^{(s_{t_c},\alpha_c)^{(0)}}(a)\right)\forall a\in D_0$. Table 14 shows the calculated power, weakness, and qualification for D_0 in Iteration one of distillation 3. The maximum $(s_{t_c},\alpha_c)^{(k+1)}$-qualification score is $\vec{q}_{D_k}=\max_{x\in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)=2$, then $\vec{D}_{k+1}=\left\{a\in D_k \middle| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a)=\vec{q}_{D_k}\right\}$, $\vec{D}_1=\{ei_5,ei_7\}$. Because $|\vec{D}_1|>1$ we proceed with another iteration in distillation three.

Iteration 2: Let $k=1$, and $(s_{t_c},\alpha_c)^{(2)}=\max_{\substack{\{\sigma(a,b)+z((s_{t_c},\alpha_c)^{(1)})<(s_{t_c},\alpha_c)^{(1)}\}\\a,b\in D_1}}\bar{\sigma}(a,b)=(S_6^P,0.24)$.

From this linguistic cutoff, we create the linguistic crisp outranking relation shown in Table 15. Then we calculate the $(s_{t_c},\alpha_c)^{(2)}$-qualifications $\left(q_A^{(s_{t_c},\alpha_c)^{(1)}}(a)\right)\forall a\in D_1$. Table 16 shows the calculated power, weakness, and qualification for $\vec{D}_1$ in Iteration 2 of distillation 3.

The maximum $(s_{t_c},\alpha_c)^{(k+1)}$-qualification score is $\vec{q}_{D_k}=\max_{x\in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)=1$, then:

$$\vec{D}_{k+1}=\left\{a\in D_k \middle| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a)=\vec{q}_{D_k}\right\}, \quad (47)$$

$$\vec{D}_1=\{ei_5\}.$$

Because $|\vec{D}_1|=1$ then $\vec{C}_3=\vec{D}_1=\{ei_5\}$ and the third distillation is completed. For the subsequent distillation, put $\overrightarrow{EI}_3=\overrightarrow{EI}_2\backslash\vec{C}_3=\{ei_1,ei_2,ei_4,ei_5,ei_7\}\backslash\{ei_5\}=\{ei_1,ei_2,ei_4,ei_7\}$ and do $n=n+1=3$.

Distillation 4

Iteration 1: Let $k=0, D_0=\overrightarrow{EI_3}=\{ei_1,ei_2,ei_4,ei_7\}$, then $(s_{t_c},\alpha_c)^{(0)}=\max_{\substack{a,b\in\overrightarrow{EI_3}\\a\neq b}}\bar{\sigma}(a,b)=(S_8^P,0)$, and $(s_{t_c},\alpha_c)^{(2)}=\max_{\substack{\{\sigma(a,b)+z((s_{t_c},\alpha_c)^{(1)})<(s_{t_c},\alpha_c)^{(1)}\}\\a,b\in D_1}}\bar{\sigma}(a,b)=(S_7^P,0.84)$. From this linguistic cutoff, we create the linguistic crisp outranking relation shown in Table 17. Then, we calculate the $(s_{t_c},\alpha_c)^{(1)}$-qualifications $\left(q_A^{(s_{t_c},\alpha_c)^{(0)}}(a)\right)\forall a\in D_0$. Table 18 shows the calculated power, weakness, and qualification for $\vec{D}_0$ in Iteration one of distillation 4. The maximum $(s_{t_c},\alpha_c)^{(k+1)}$-qualification score is $\vec{q}_{D_k}=\max_{x\in D_k} q_{D_k}^{(s_{t_c},\alpha_c)^{(k)}}(x)=1$, then $\vec{D}_{k+1}=\left\{a\in D_k \middle| q_{D_k}^{(s_{t_c},\alpha_c)^{(k+1)}}(a)=\vec{q}_{D_k}\right\}$, $\vec{D}_1=\{ei_7\}$. Because $|\vec{D}_1|=1$ then $\vec{C}_4=\vec{D}_1=\{ei_7\}$ and the third distillation is completed. For the subsequent distillation, put $\overrightarrow{EI}_4=\overrightarrow{EI}_3\backslash\vec{C}_4=\{ei_1,ei_2,ei_4,ei_7\}\backslash\{ei_7\}=\{ei_1,ei_2,ei_4\}$ and do do $n=n+1=4$.

The remaining steps for the next descending distillations and the ascending distillation steps are processed in the same way. After completing the descending and ascending distillations, we got two complete preorders whose intersection creates the final ranking of the alternatives.

Figure 4 depicts the two preorders (descending and ascending distillations) calculated with the distillation procedure. In the descending preorder (Fig 4. a), there is an equivalence class in the first rank with the environmental impacts ei_1,ei_2,ei_4,ei_7, followed by EI ei_5 in the second rank, and at the last rank, there is an equivalence class with the environmental impacts ei_3 and ei_6. Meanwhile, the ascending distillation (Fig 4. b) is more granulated with ei_1 in the first rank, followed by $ei_4\succ ei_7\succ ei_5\succ ei_2$, and at the last rank are ei_3 and ei_6.

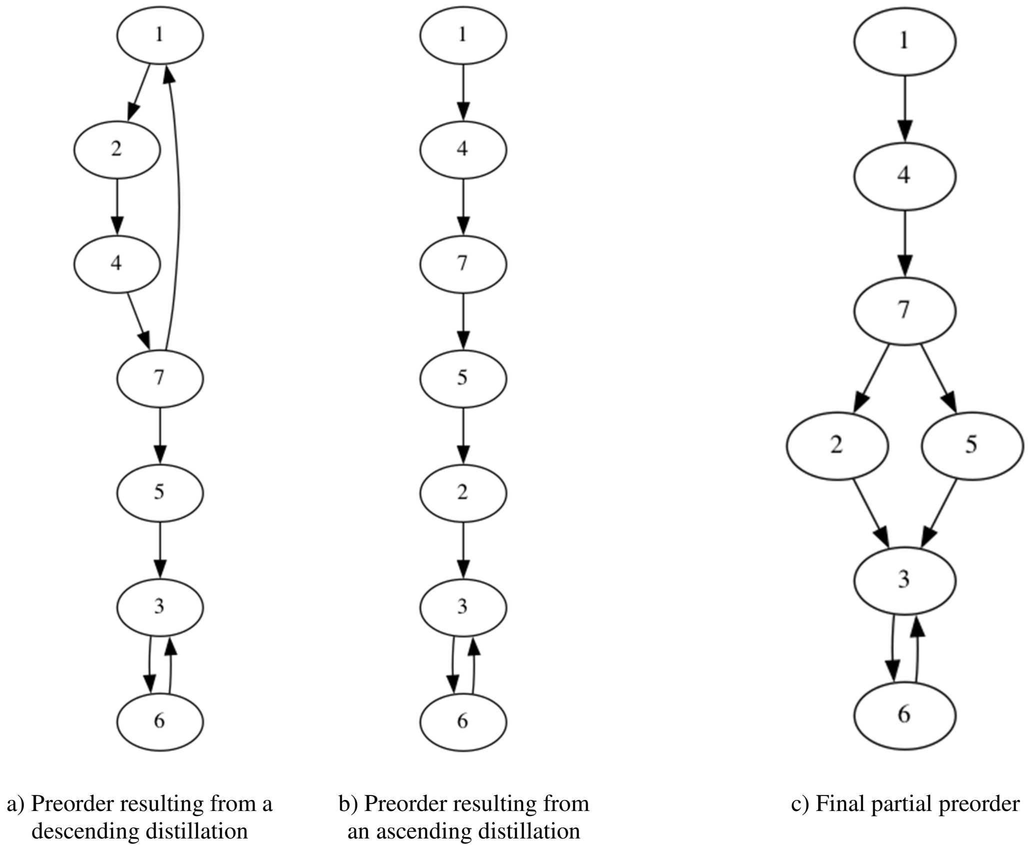

Fig. 4. Graphical representation of the preorders

Figure 4. c depicts the final preorder resulting from the intersection of the two preorders.

The final partial preorder follows a decreasing order of preferences, meaning that environmental impacts at the top are more significant than those at the bottom.

Hence, the final rank suggests that the ei_1 is the most significant environmental impact that is the interaction between action a_1 and factor f_2; in the second position is ei_4, that is the interaction between action a_2 and factor f_4; in the third position is ei_7 that is the interaction between action a_4 and factor f_1; in the fourth position are ei_2 and ei_5 that are the interactions between a_1 and factor f_3, and a_3 and factor f_1 respectively; it should be noted that although ei_2 and ei_5 are in the same ranking, they are incomparable. Thus, more analysis should be made for these two actions; finally, in the last rank, there is an equivalence class with ei_3 and ei_6 that suggests that both are indifferent to each other.

5 Conclusions

This paper aimed to develop a linguistic extension of the ELECTRE III method that allows solving instances of the multicriteria ranking problem with input data defined in heterogeneous contexts.

The new proposal fuses the heterogeneous information into 2-tuple linguistic values, allowing the DM to provide their preferences using diverse expression domains, such as numerical domain, interval-valued domain, and linguistic domain, according to the nature and uncertainty of the decision criteria, and their level of knowledge and experience.

Consequently, the new method is appropriate to integrate quantitative and qualitative criteria and uncertain information into the elements of the multicriteria model. In the modeling process of the ELECTRE III linguistic method, concordance, discordance, and credibility indices are proposed to consider linguistic inputs and outputs. Also, a linguistic difference function is stated to compute the linguistic difference between a pair of 2-tuple linguistic values. The output of the linguistic difference function is the input of the linguistic concordance and discordance indices.

Therefore, the linguistic extension of the ELECTRE III method offers good quality interpretability and understanding throughout the decision-making process in instances of the multicriteria ranking problem where there is heterogeneous data, as demonstrated in the illustrative example presented in this document.

The proposed methodology is applicable to real-life situations that involve decision-making with multiple conflicting criteria. This methodology can be used for various purposes such as project selection, supplier selection, job candidate evaluation, product design, environmental policy, and more.

When making decisions in contexts that involve diverse perspectives or input data from various sources, applying the linguistic ELECTRE III method can significantly impact decision-making in business, government, or social environments. For example, it can enhance the consideration of decision-maker preferences, improve transparency and accountability, facilitate cross-sector collaboration, and enable adaptation to dynamic environments.

The linguistic ELECTRE III can help organizations and policymakers navigate complex situations, manage uncertainty, and make more informed and equitable decisions in various business, government, and social environments. It provides a systematic and structured approach to decision-making in diverse contexts.

Soon, we plan to develop a linguistic extension of the ELECTRE III method for a collaborative group of DMs, and a hierarchical linguistic extension of the ELECTRE III method. Also, it is contemplated to carry out more real-world applications of the multicriteria ranking problem using our proposal.

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**Corresponding author is Juan Carlos Leyva López.*