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Water Stress Challenges: Mathematical Modeling of Water Resource Management

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Abstract. Water resource management is an important issue that involves several factors such as economics, social, politician, among others, for its adequate administration. Water can be classified according to its usage purposes since it is used for human consumption, industrial usage, agriculture, etc. Thus, correct strategies to manage this vital liquid are essential for its effective use. This paper studies water management from a mathematical optimization approach by considering factors and constraints that may suit real-world conditions. The proposed mathematical model is based on the classical transportation problem, which is well-known in the literature. We perform an empirical evaluation of the proposed model using off-the-shelf optimization software over a set of proposed instances, and the results show the feasibility of the proposal. Finally, we discuss the faced challenges in the research and possible future research directions that may help the management of water resources from a computational approach.

Keywords. Water stress, water management optimization, transportation problem, mathematical optimization.

1 Introduction

Water, a vital element for the sustenance of living organisms, faces relentless exploitation due to the current demands and circumstances of humanity. The imperative needs of various economic sectors drive the excessive use of this indispensable resource. According to [19], global water consumption has been on the rise at approximately 1% annually over the past four decades, and this trend is anticipated to continue until 2050. Despite the inherent renewability of water, its consumption extends beyond human needs, encompassing commercial, industrial, agricultural, livestock, and energy production activities, causing the depletion of water sources, surpassing natural renewal by the hydrological cycle and causing water stress [4].

Efficient water resource management faces numerous limitations and uncertainties, making decision-making very challenging. For example, water is extracted from different sources such as basins, rivers, lakes, etc., and its processing varies depending on the sector in which it is going to be used, so in the end, you have different types of water. That is, the water used for human consumption has different characteristics than that used for irrigation or industry. Furthermore, the volume of water to be used must be adapted based on the different demands linked to each specific sector (i.e., agricultural, industrial, etc.), population density, geographical conditions, and climate change [17, 27], not to mention that all these aspects require a lot of bureaucracy, which further complicates the management of water resources.

The study of the problem of water management using computational tools has been going on for years, for example, the estimation of hydrogeological parameters [8, 10] to establish environmental policies, studies on saltwater intrusion into coastal aquifers by using evolutionary algorithms [1, 3], pollution management in hydrographic basins [25], and optimal water allocation for crops and irrigation [16], are topics of interest in the computational world. In addition, alternative methodologies involving the modeling of water management issues using a spectrum of tools have arisen. For instance, in [11] was developed a dynamic model leveraging the expertise of various domain experts, facilitating the selection of optimal water management activities to mitigate water shortages. In [20] was introduced a hydrological and system dynamics model specifically designed to analyze five distinct scenarios about industrial, agricultural, and domestic water use. [28] contributed to the field by enhancing the Water Resources Ecological Footprint, with a particular emphasis on regional distinctions.

In a different vein, in [30] was introduced a stochastic multi-criteria decision-making framework for Water Resource Management, explicitly considering the challenges posed by uncertainty. In [29] was presented a synthesis of key concepts and categories related to urban drought, elucidating strategies to enhance public awareness, promote flexibility, optimize water management efficiency, ensure reliable and integrated urban water supply, invest in scientific research and strengthen international cooperation. In [24], was addressed Water Resource Management, specifically targeting irrigation systems through the application of algorithms to calculate limits. In the domain of agricultural water management, in [21] was proposed a generalized spatial fuzzy strategic planning approach, incorporating multi-criteria decision-making.

A strong trend is the optimal design of water distribution systems, whether to improve distribution strategies, pipe rehabilitation, water quality, avoid leaks, optimize the operation of pumps, and also the occurrence of water contamination [14]. Also, parallel evolutionary algorithms have been proposed for similar approaches to optimizing the network design for water distribution [2]. However, efforts have focused on finding the best system design (at a local level) that maximizes the robustness of the network and at the same time is cost-effective, but the problem of water management is not addressed in a broad context, for example, the challenge of balancing water consumption to promote the replenishment of water resource sources. Furthermore, if the restrictions mentioned in the previous paragraph (i.e. different supply and demand sources, different types of water, and other restrictions) are added, the problem becomes more difficult to solve. Motivated by the lack of such studies in the literature, we address the challenge of Water Resource Management, where multiple types of source water resources are involved. Besides, other constraints that may suit real-life conditions are considered.

To propose a computational solution, it is necessary to mathematically model the problem. In the state of the art, different approaches could be used to model the problem. However, given that the problem in its simplest form consists of taking water from supply points to demand points, we consider that an approach based on the transportation problem could be a good choice. The Transportation Problem is traditionally linked to the operations research literature [7], which can be seen as the simplification of the objective of minimizing the costs of the carrier that moves certain cargo from one or more origins to their corresponding destinations to satisfy demand.

In this work, we propose a mathematical model as an extension of the transportation problem, where a bipartite graph is established that considers supply nodes, demand nodes, and an associated cost of water transportation. The aim is to minimize the cost of transportation, but our approach does not end there, since as mentioned above, different factors complicate the efficient management of water resources; these factors must be considered in our model to propose solutions that are more in line with reality. It is for these reasons that we also incorporate different restrictions that prevent excessive water use. This is of vital importance since it would allow the natural renewal of water resource sources. All these factors make the problem computationally more interesting. Detailed information on restrictions is set out in Section 3.

To test the proposal, we designed and coded a generator of feasible instances that were solved using the proposed mathematical model; however, due to approached restrictions, there are limits in the size of the instances that can be generated in feasible computational time. It is important to note that the objective of the work is to show a mathematical model that allows the management of water resources considering restrictions attached to reality, so the application of metaheuristic approaches is outside the scope of this work. However, in Section 5, we establish the necessary guidelines to address the problem through metaheuristic optimization, which is why we frame it as future work.

To the best of our knowledge, there is no approach similar to the one proposed in the literature, so the results reported in the present work provide valuable knowledge to experts in the field of computational sciences and water resources management. Providing an approach that helps make informed decisions based on data.

The rest of the paper is organized as follows: Section 2 shows the background about water resources and mathematical optimization. Specifically linear programming and the transportation problem. In Section 3, the proposal is described in detail, which consists of a Mixed Integer Quadratically Constrained Program (MIQCP). This mathematical model takes as basis the classical transportation problem. Besides, the assumptions and limitations of this mathematical model are discussed. Section 4 describes the experimental design, the followed methodology, and the faced challenges in generating the instances. Section 5 performs an analysis of the obtained results. Finally, Section 6 states the conclusions and discusses the possible future work directions of this work.

2 Background

2.1 Water Resources

Various types of water resources originate from natural sources and serve human, agricultural, or industrial purposes. [4] classify the water resources into two main categories: surface water and groundwater. Surface water includes water flows that traverse the earth's surface (such as rivers). It encloses bodies of water gathered in naturally occurring or human-made depressions, like dams and lakes, as well as in periodically or permanently flooded areas, such as swamps and wetlands. Groundwater consists of rainwater retained in impermeable soil. This resource holds significance as it functions both as a versatile natural water storage and a distribution network for a country. The term physical water stress refers to the ratio of water usage to available water, and it is determined by a combination of various factors [19]. The global rise in water scarcity is a consequence of escalating physical water stress, impacting regions worldwide. It's worth noting that the quality and availability of these water resources vary based on factors like geographic location, land use practices, climatic conditions, population growth, infrastructure development, over-extraction, and regulatory policies.

2.2 Water Management

As outlined in The 2030 Agenda for Sustainable Development [15] presented by the United Nations, there are 17 established Sustainable Development Goals. The sixth goal, known as SDG 6, aims to guarantee the accessibility and sustainable supervision of water and sanitation, along with the sustainable handling of water resources, water quality, integrated water resources management, water-related ecosystems, and the creation of a conducive environment. The 2023 United Nations World Water Development Report [19] asserts that the demand for water in agriculture is primarily influenced by irrigation, with variations dependent on various determining factors. Another crucial factor to consider is the per capita water availability, which has been diminishing due to the growth rates in population. Therefore, efforts have been made to implement initiatives aimed at developing alternatives that streamline decision-making and enhance the prediction of diverse factors. The goal is to optimize water management with greater efficiency. Models can help to represent the interactions between these factors and their complex interactions. As highlighted by [9], mathematical models are primarily categorized into two main types:

simulation-based or optimization-based models. The last one can be further sub-categorized into three distinct groups: conflict resolution models, water resources planning models, and models addressing water availability and demand diagnosis. The last category helps to estimate the water availability and compare it with the water demand to find optimal strategies for meeting these demands efficiently. Although these models can provide important information, the final decisions rest with the stakeholders.

2.3 Linear Programming (LP) and Mathematical Optimization

LP [13] or lineal optimization is a mathematical method for solving optimization problems where the objective is to optimize a linear function under constraints represented as linear equalities and inequalities. The main objective is to find the best combination of all the variables that satisfy all the constraints for the problem to determine a way to achieve the best outcome (for example, determine the lowest cost). The following equations (1)-(5) represent the standard form of a LP [26]:

$$\min \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad (1)$$

subject to

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1, \quad (2)$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2, \quad (3)$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m, \quad (4)$$

$$x_1, x_2, \cdots, x_n \geq 0, \quad (5)$$

where the b_i's, c_i's, and $a_{i,j}$'s are fixed real constant numbers, and the x_i's are real numbers to be determined which are called decision variables. Generally, a classical LP satisfies the following conditions: the variables of the problem must be non-negative, the objective function should express a linear combination of variables through a linear function, and the constraint set must consist of linear equations or inequalities.

The model should adapt to the problem by considering all the specific variables and constraints it must fulfill. LP has been widely used in different problems such as the routing selection problem [18], the transportation problem [22], and the supply-chain problem [23].

In addition to LP, there are other practical mathematical optimization approaches for scenarios that cannot satisfy linearity. Mixed Integer Programming (MIP) is a mathematical optimization approach based on the general principles of LP, but its decision variables consist of both integer and real values. The classification of MIP problems depends on the nature of the objective function and constraints. The problem is called a Mixed Integer Linear Program (MILP) when the objective function and constraints are linear. However, if the objective function includes a quadratic term, it is called a Mixed Integer Quadratic Problem (MIQP). In addition, a model is said to be a Mixed Integer Quadratically Constrained Program (MIQCP) [31] if it contains constraints with quadratic terms, regardless of the form of the objective function.

2.4 The Transportation Problem

The transportation problem stands as an essential optimization problem widely investigated in the field of operations research. Its main application lies in the efficient distribution of goods from a predefined set of source vertices to a designated set of destination vertices, with the general objective of minimizing the associated costs. As a fundamental element in various economic, social, and market scenarios, the Transportation Problem assumes a critical role in optimizing logistics processes [7].

Formally, the Transportation Problem is stated as follows.

Consider the set of supply vertices, denoted as $V = \{v_1, v_2, \cdots v_n\}$, and a supply function $S : V \to \mathbb{R}^+$, where each vertex $v_i \in V$ is endowed with the capacity to transport up to $S(v_i)$ units of goods.

Let $U = \{u_1, u_2, \cdots u_m\}$ represent the set of demand vertices corresponding to sites necessitating the delivery of goods. The demand function is defined as $D : U \to \mathbb{R}^+$, specifying that each vertex $u_j \in U$ requires the fulfillment of a demand amounting to $D(u_j)$.

The classical transportation problem can be formally characterized through Expressions (6)-(9):

$$\min \sum_{v_i \in V} \sum_{u_j \in U} c_{i,j} x_{i,j} \tag{6}$$

subject to

$$\sum_{v_i \in V} x_{i,j} \geq D(u_j) \qquad \forall u_j \in U, \tag{7}$$

$$\sum_{u_j \in U} x_{i,j} \leq S(v_i) \qquad \forall v_i \in V, \tag{8}$$

$$x_{i,j} \in \mathbb{R}^+ \qquad \forall v_i \in V, \forall u_j \in U. \tag{9}$$

The equations presented make up a Linear Programming (LP) formulation, where $c_{i,j}$ is the associated cost of transporting one unit of goods from source vertex v_i to demand vertex u_j, and $x_{i,j}$ denotes the quantity of goods units transported from v_i to u_j. Consequently, if $x_{i,j}$ goods units are transported from v_i to u_j, the corresponding cost is $c_{i,j}x_{i,j}$.

In this LP framework, (6) is the objective function to minimize the total transportation cost from source to demand vertices. The constraints stated in (7), ensure the satisfaction of demand for each u_j, while the (8) constraints state that the total goods shipped from the origin vertex v_i do not exceed the available quantity. Finally, the expression (9) defines the decision variables. It is important to note that this model assumes viability, that is, total supply equals or exceeds total demand, as established in the following equation:

$$\sum_{u_j \in U} D(u_j) \leq \sum_{v_i \in V} S(v_i). \tag{10}$$

It is well-known that the classical transportation problem can be solved efficiently using LP techniques [5, 6], but real-world scenarios often require more complex constraints, which pose challenges in solving these types of problems.

In the next section, we present a model based on the transportation problem that abstracts the problem of Water Resources Management. However, it presents additional restrictions that may arise in real-world scenarios, which complicate the optimization problem and increase computational demand.

3 Proposal

In this section, we propose a Mixed Integer Quadratically Constrained Program (MIQCP) specifically designed to address the Water Resources Management problem. Rooted in the fundamental principles of the transportation problem, this model incorporates additional constraints essential to address the complications inherent to the problem studied. The integration of these constraints enriches the model, which resembles real-world conditions.

3.1 Proposed Mathematical Model

In the context of water resources management, we propose to address the challenge of water distribution by modeling a scenario in which water is supplied from different water sources $V = \{v_1, v_2, ..., v_n\}$ to different demand locations $U = \{u_1, u_2, ..., u_n\}$, considering that each source and demand location manages a different type of water included in the set $K = \{k_1, k_2, \cdots, k_p\}$, where p is the number of types of water, and k_i the type of water.

In addition, we establish a supply function S, a demand function D, and a function $T : V \cup U \rightarrow K$, the latter guaranteeing that each source vertex $v_i \in V$ can supply exclusively to the demand vertices within the set $\{u_j \in U : T(u_j) = T(v_i)\}$. That is, a demand vertex that requires a specific type of water k_i, can only be satisfied by source vertices that supply the same type of water. Since in a real context, processed water for the industry would not be sent to a place for human consumption. This delineation of water types and the associated constraints through the function T introduces an added layer of complexity to the classical transportation problem, catering to the nuanced requirements of the problem considered in this study.

In this scenario, the method of water transportation is inconsequential; That is, we ignore the specific mode of transportation and instead introduce the term “carriers”, which fulfill the function of transporting water units from the source vertices to the demand vertices, establishing a cost associated with said transportation, which can be different between carriers. This cost is a crucial factor since it is a function of all the aforementioned variables, and quantifying and optimizing it becomes essential in our study.

To elaborate, we define a set of carriers, denoted as $C = \{1, 2, 3, \ldots, |C|\}$, where each carrier $l \in C$ sets an associated cost $c^l_{i,j}$ to the transport of a unit of water from a source vertex $v_i \in V$ to the demand vertex $u_j \in U$. To achieve load balancing between carriers, each operator $l \in C$ is assigned a capacity $L(l) \in \mathbb{N}$, which represents the maximum number of source vertices that it can drive. In line with our general objective, based on this mathematical model we seek to minimize the cost of transporting water satisfying all demands.

Given the multitude of constraints and variables involved, we provide a concise summary of the key assumptions underlying the problem at hand for clarity and precision:

1. All carriers can deal with any type of water, any source vertex, and any demand vertex.
2. It is established that there is sufficient capacity among carriers to operate at all origin vertices. See the following equation:

$$\sum_{l \in C} L(l) \geq |V| \qquad \text{holds.} \tag{11}$$

3. It is vitally important to consider that for each type of water, the total supply equals or exceeds the demand. This is stated in the following equation:

$$\sum_{u_j \in U : T(u_j) = k_t} D(u_j) \leq \sum_{v_i \in V : T(v_i) = k_t} S(v_i). \tag{12}$$

It holds $\forall k_t \in K$. Equations (13)-(20) introduce a mathematical model for the described problem:

$$\min \sum_{l \in C} \sum_{v_i \in V} \sum_{u_j \in U} c^l_{i,j} x^l_{i,j} \tag{13}$$

subject to

$$\sum_{l \in C} \sum_{\substack{u_j \in U: \\ T(v_i) \neq T(u_j)}} x^l_{i,j} = 0 \qquad \forall v_i \in V, \tag{14}$$

$$\sum_{v_i \in V} \sum_{l \in C} y_{l,i} x^l_{i,j} \geq D(u_j) \qquad \forall u_j \in U, \tag{15}$$

$$\sum_{l \in C} \sum_{u_j \in U} x^l_{i,j} \leq S(v_i) \qquad \forall v_i \in V, \tag{16}$$

$$\sum_{v_i \in V} y_{l,i} \leq L(l) \qquad \forall l \in C, \tag{17}$$

$$x^l_{i,j} \in \mathbb{R}^+ \qquad \forall (v_i \in V, u_j \in U, l \in C), \tag{18}$$

$$y_{l,i} \in \{0, 1\} \qquad \forall l \in C, \forall v_i \in V, \tag{19}$$

where $x^l_{i,j}$ is the amount of water units to be shipped from v_i to u_j through carrier l, and:

$$y_{l,i} = \begin{cases} 1, & \text{if carrier } l \text{ is assigned to vertex } v_i, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

In this model, the objective function (13) seeks to minimize transportation costs, encompassing all carriers. Constraint (14) dictates that demand vertices are exclusively supplied by source vertices with matching water types. Ensuring the satisfaction of water demands, constraint (15) plays a crucial role.

To prevent excessive extraction and potential stress on water bodies, constraint (16) curtail the amount of water drawn from each source vertex to within its available capacity. Pertinently, these constraints hold significant implications in the context of water supply.

Meanwhile, constraint (17) safeguards against exceeding carrier capacities when attending to source vertices. Finally, the decision variables are defined and described through expressions (18)–(20).

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3.2 Water Resources Optimization through Mathematical Optimization

To show the feasibility of the proposal, we show an example to clarify how the mathematical model works.

3.2.1 Objective Function Evaluation

Next, the process of evaluating the objective function is shown. That is, to find the values of the decision variables that optimize the function and simultaneously satisfy the constraints.

First, we establish the scenario to optimize. Visually, this can be represented through a bipartite graph where the set of supply nodes V, the set of demand nodes U, the set of types of water K, and the set of carriers C are established. It is important to remember that each carrier $l \in C$ establishes a cost $c^l_{i,j}$ for transporting a unit of water from $v_i \in V$ to $u_j \in U$, and a capacity $L(l)$ of supply nodes that it can attend. Furthermore, each scenario must satisfy (11) and (12), as well as the constraints (14)–(20) imposed on the model, this allows the problem to have a feasible solution. However, this feature makes the optimization problem difficult since a solution must be in the feasible space. Fig. 1 shows an example for an instance with $|V| = 6$ source vertices, $|U| = 4$ demand vertices, $|K| = 2$ types of water, and $|C| = 3$ carriers.

Concerning the capacity of the carriers $L(l)$, the supply water units $S(v_i)$ and the demand water units $D(u_j)$, these are established randomly but complying with the restrictions (11), (12) and (14) to find a scenario with feasible solutions. Finally, transportation costs are also randomly assigned to a range of positive numbers. The complete instance of this scenario can be consulted in the link provided in the Test Instances section.

To calculate the objective value using (13), the transportation costs of the instance above are used, which are summarized in Table 1, along with the water units $x^l_{i,j}$ obtained by the mathematical model to optimize the problem established in Fig. 1, for each carrier $l \in C$ and its associated transportation costs $c^l_{i,j}$. Using the values from Table 1, the objective value obtained is 65.

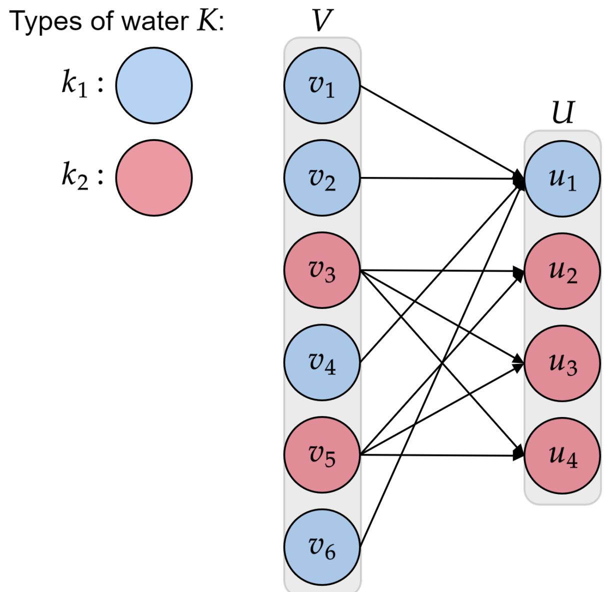

Fig. 1. Example scenario: $|V| = 6$, $|U| = 4$, $|K| = 2$ and $|C| = 3$

Table 1. Solution obtained by the model

Carriers	Transportation cost	Water units
$l = 1$	$c^1_{3,2} = 1$	$x^1_{3,2} = 8$
	$c^1_{3,4} = 3$	$x^1_{3,4} = 7$
	$c^1_{5,3} = 3$	$x^1_{5,3} = 9$
$l = 3$	$c^3_{6,1} = 1$	$x^3_{6,1} = 9$

Fig. 2 shows the optimal solution found using the mathematical model. We can verify this solution meets all established restrictions.

4 Experimental Design and Results

4.1 Test Instances

To assess the robustness of the proposed model, we formulate 30 instances that satisfy the constraints described in Section 3.1. Table 2 presents these instances along with their respective parameters, where $|V|$ and $|U|$ are the number of supply and demand vertices respectively, $|K|$ is the number of water types, and $|C|$ is the number of carriers. The design of these instances is deliberate and features a gradual escalation of difficulty, either by adding supply and

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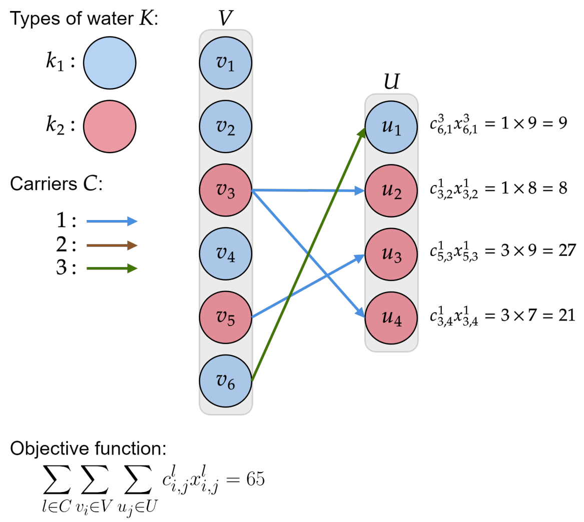

Fig. 2. Solution for the example scenario: $|V| = 6$, $|U| = 4$, $|K| = 2$, and $|C| = 3$

demand vertices, variations in water types, or an increase in the number of carriers.

Instances 1–5 represent the simplest cases, featuring 2 types of water and 2 carriers. The number of supply nodes $|S|$ is twice that of demand nodes $|D|$ in each instance. In contrast, Instances 6–10 mirror Instances 1–5, with the number of demand nodes $|D|$ being half that of supply nodes $|S|$. This deliberate design allows us to evaluate the model's performance under varied scenarios with unequal supply and demand nodes.

Instances 11–15 and 16–20 present a considerable increase in problem difficulty. Here, the number of water types grows by increments of 5, ranging from 5 to 25. Simultaneously, the number of carriers increases by 10 for each instance, starting at 10 and concluding at 50. Instances 11–15 and 16–20 are mirror instances, enabling a comprehensive assessment of the model's adaptability to varied configurations.

In this study, the most challenging scenarios are Instances 21–25 and 26–30, designed to push the model's limits. The complexity is heightened by increasing the number of water types by 5, from 30 to 50. Additionally, the number of carriers increases by 50, starting at 50 and concluding at 250 for each instance. Instances 21–25 and 26–30 are mirror instances, providing a thorough exploration of the model's capabilities under difficult conditions.

Table 2. Test instances configuration

Instance	$\|V\|$	$\|U\|$	$\|K\|$	$\|C\|$
1	1	2	2	2
2	2	4	2	2
3	3	6	2	2
4	4	8	2	2
5	5	10	2	2
6	2	1	2	2
7	4	2	2	2
8	6	3	2	2
9	8	4	2	2
10	10	5	2	2
11	15	30	5	10
12	20	40	10	20
13	30	60	15	30
14	40	80	20	40
15	50	100	25	50
16	30	15	5	10
17	40	20	10	20
18	60	30	15	30
19	80	40	20	40
20	100	50	25	50
21	150	350	30	50
22	200	300	35	100
23	250	250	40	150
24	300	200	45	200
25	350	150	50	250
26	350	150	30	50
27	300	200	35	100
28	250	250	40	150
29	200	300	45	200
30	150	350	50	250

Finally, for these instances, water units for both supply and demand vertices were randomly assigned within the following ranges: $S(v_i) \in [1000, 5000]$, $D(u_j) \in [100, 500]$. The capacity of carriers was set randomly within the range $L(l) \in [1, |V|]$. Finally, transportation costs associated with each carrier were randomly established within the range $c^l_{i,j} \in [100, 1000]$. The complete instances can be consulted here[1].

4.2 Parameter Configuration

The mathematical model was implemented in the Python programming language by using the off-the-shelf optimization software Gurobi v10. The Gurobi software implements different mathematical optimization algorithms, such as LP algorithms like Simplex and Barrier, Branch-and-Bound for MIP problems, among others [12].

All the experiments were run on a computer with a Windows 11 OS, 40 GB of RAM, and an Intel i7-10750H processor.

For the mathematical model, we tested three different relaxations available in the Gurobi software: Primal Simplex (PS), Dual Simplex (DS), and Barrier (B). Table 3 shows the results obtained from the experimentation. From this table, **OPT** refers to the optimal solutions, whereas **PS** $t(s)$, **DS** $t(s)$, and **B** $t(s)$ refer to the running time per instance for each relaxation method.

5 Analysis and Discussion of Results

In this section, we explore challenges in optimizing the proposed model with added types of water and carriers, impacting solution space, instance generation, and resolution dynamics.

The obtained results showcase the potential applicability in real-world scenarios. The results reported in Table 3 affirm the feasibility of optimizing our proposal using a mathematical optimization approach. In all the cases, the optimal solutions were found. For this experimentation, we can appreciate that using different relaxation techniques does not change radically the running time. However, for bigger instances, we could not ensure this. Through experimentation, we noticed that the running time of different relaxation algorithms can change drastically for some instances with $|V| > 600$. Nevertheless,

Table 3. Optimization results for each instance, reporting its optimal value **OPT** and the execution time for each strategy measured in seconds $t(s)$

Instance	OPT	PS $t(s)$	DS $t(s)$	B $t(s)$
1	235,044	0.003	0.034	0.036
2	405,326	0.001	0.002	0.000
3	656,886	0.004	0.004	0.013
4	793,611	0.003	0.004	0.004
5	489,604	0.003	0.003	0.007
6	96,280	0.002	0.001	0.001
7	227,303	0.002	0.000	0.000
8	85,908	0.004	0.005	0.004
9	249,383	0.003	0.000	0.001
10	304,674	0.004	0.004	0.006
11	1,310,305	0.027	0.022	0.081
12	1,290,357	0.034	0.031	0.050
13	2,180,535	0.081	0.086	0.083
14	2,527,523	0.162	0.146	0.167
15	3,582,503	0.270	0.297	0.283
16	705,887	0.019	0.029	0.033
17	703,887	0.022	0.016	0.017
18	1,034,468	0.068	0.062	0.066
19	1,308,086	0.140	0.159	0.150
20	1,680,449	0.296	0.303	0.299
21	10,854,584	4.454	3.864	5.518
22	9,084,413	11.084	9.272	13.205
23	7,580,286	18.027	15.151	21.556
24	6,212,880	22.829	19.389	26.393
25	4,544,234	22.190	22.589	22.874
26	4,744,219	5.067	4.297	5.902
27	5,871,442	11.502	10.102	13.886
28	7,716,824	17.815	15.057	21.407
29	9,287,937	19.743	18.116	20.121
30	10,517,914	22.349	20.405	23.261

[1] https://github.com/alex-cornejo/WaterManagement-ComSis

we could not include experimentation for bigger instances due to the practical issues discussed below.

The escalating complexity introduced by including more types of water and carriers imposes significant challenges on the problem. The imposed constraints not only shape the feasible solution space but also impact both the instance generation process and the optimization procedure.

Through empirical experimentation, we observed that the water type constraint poses a more intricate challenge for the optimization process than the carrier capacity constraint. This complexity is evident in the increased time required to resolve instances. Conversely, with a growing number of carriers, the memory requirements for processing instances also surge. Each carrier, having an associated transportation cost expressed in a cost matrix, contributes to the memory load. For instance, if there are 200 carriers, there would be 200 distinct cost matrices per instance stored in memory. Furthermore, the size of the matrix ($|V| \times |U|$) is contingent on the number of supply and demand nodes. Therefore, spatial complexity becomes a critical consideration for both instance generation and optimization.

These challenges could be effectively addressed by adopting other optimization techniques, such as evolutionary computation of metaheuristics. For instance, solution representation in metaheuristics could involve a set of genes encoding the assignment of water types to carriers alongside other pertinent parameters. Designing crossover and mutation operators respecting problem constraints ensures the generation of feasible solutions. Selection operators favoring diversity and exploration of the search space can be implemented. Strategies can be integrated to handle specific constraints on the type of water, such as sanctions in the objective function for non-compliance, or remedial mechanisms for infeasible solutions.

Metaheuristics also allows for the consideration of parallelism or distribution strategies to optimize execution time, which is particularly crucial for large optimization problems. Lastly, while a comprehensive study of the computational complexity of the problem would be valuable, this aspect will be rigorously addressed in future work.

6 Conclusion and Future Work

This paper introduces an innovative approach to tackling the global water stress challenge through the application of mathematical optimization, framing the Water Resources Management problem. We performed this by proposing a mathematical model, specifically an MIQCP. The proposed mathematical model is akin to the classical transportation problem, which is well-known in the field of operations research. Then, we used off-the-shelf optimization software to test the mathematical model over a set of proposed instances that consider restrictions of possible real scenarios.

The results serve as a robust affirmation, supporting the effectiveness and utility of the proposed model in addressing optimization challenges related to water use. These findings underscore the model's practical applicability and its efficacy in solving real-world problems associated with Water Resource Management optimization.

Our study reveals that the inclusion of multiple water types introduces increased complexity. Instances with over 50 different water types proved more intricate, necessitating a scaling of computational resources. This adaptation becomes crucial to overcome model limitations and enhance the likelihood of finding viable solutions, hinting at the potential for specialized optimization strategies such as evolutionary computation and heuristics/metaheuristics.

In future work, we will focus on refining the model to increasingly align it with real-world scenarios, which will involve deep analysis of water management information. The complexity of the problem will also be rigorously studied, together with the possibility of improving the model by looking to linearize the constraints or propose new mathematical models with practical advantages. Finally, other strategies may be considered, particularly the implementation of evolutionary computing or heuristic/metaheuristic approaches.

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