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Intuitionistic Epistemic Logic with Distributed Knowledge

Ryo Murai[1], Katsuhiko Sano[2]

[1] Hokkaido University,
Japan

[2] Hokkaido University,
Faculty of Humanities and Human Sciences,
Japan

ryo.murai11@gmail.com, v-sano@let.hokudai.ac.jp

Abstract. We develop intuitionistic epistemic logics with distributed knowledge, which is more general than a logic proposed by (Jäger & Marti 2016) in that a distributed knowledge operator is parameterized by a group of agents. Specifically, we present Hilbert systems of intuitionistic K, KT, KD, K4, K4D, and S4 with distributed knowledge. The semantic completeness of the logics with regard to suitable Kripke frames is shown by modifying the standard argument of the semantic completeness of classical distributed knowledge logics via the concept of pseudo-model. We also present cut-free sequent calculi for the logics, based on which we establish Craig interpolation theorem and decidability.

Keywords. Intuitionistic logic, epistemic logic, distributed knowledge.

1 Introduction

'Distributed knowledge' is one of the notions of group knowledge studied in multi-agent epistemic logic [6, 18]. A typical example of distributed knowledge is the following: a group consisting of a and b has distributed knowledge of a fact q when a knows that $p \to q$ and b knows that p. According to [1, Section 1], "distributed knowledge is the knowledge of a third party, someone 'outside the system' who somehow has access to the epistemic states of all the group members". Fagin et al. [6, p. 3] stated as an intuitive description for distributed knowledge "a group has distributed knowledge of a fact φ if the knowledge of φ is distributed among its members, so that by pooling their knowledge together the members of the group can deduce φ". At first sight, the latter description seems clearer than the former. Ågotnes and Wáng [1] state, however, that the above intuitive description by Fagin et al. [6, p. 3] is inappropriate by an illustrative example given in [1, Section 1].

Formally, distributed knowledge is expressed as a modal operator D_G, parameterized by a finite group G of agents and the satisfaction of $D_G\varphi$ at a state w is defined as: φ holds at all states v such that v can be reached in a single step from w for all agents in G, i.e., wR_av for all agents $a \in G$, where R_a is a binary relation on the states. As for the model-theoretic study of distributed knowledge, we can cite [1, 25, 10, 28]. Proof-theoretic study is relatively less active, but there have been proposed several sequent calculi [12, 23, 11, 19]. However, those cited here are all on the basis of classical logic.

Not to mention distributed knowledge, epistemic logic as a whole has been studied mainly in the classical setting. However, several kinds of intuitionistic epistemic logics have been proposed from different perspectives. Several philosophical logicians have proposed intuitionistic epistemic logics [31, 24, 3] for the sake of analysis of Fitch's knowability paradox [7], from the verificationist point of view.

Another kind of intuitionistic epistemic logic [14] is proposed for the analysis of distributed computing in the sense of [13, 26]. Also, [27] develops an intuitionistic epistemic logic from the

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game-theoretical point of view. The intuitionistic aspect of the logic is required for describing the property of asynchronous communication among agents in distributed computing.

Jäger and Marti [15] formulate intuitionistic epistemic logic with distributed knowledge for the first time, as far as the authors know, and prove semantic completeness of Hilbert systems of intuitionistic $\mathbf{K}$ and $\mathbf{KT}$ with distributed knowledge. Logics we investigate in the present paper is basically based on theirs, but differs in the following respects: firstly, in our logics, distributed knowledge operator is parameterized by a group, i.e., a subset of whole agents, while [15] deals with only distributed knowledge for the whole agents. Secondly, we handle more axioms than [15], that is, we propose intuitionistic $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{K}4$, $\mathbf{K}4\mathbf{D}$, and $\mathbf{S}4$ with distributed knowledge. One point to note here. Axioms (K), (T), and (4) in our logics are simply a D_G-version of the respective axioms in the basic modal logic.

However, our axiom (D) is restricted to a single agent (i.e., $\neg D_{\{a\}}\bot$). This is because seriality for each R_a is generally not preserved under taking intersection among a group (refer to [2]), while reflexivity and transitivity are always preserved. As for proof of the semantic completeness, we adopt a more standard method via the concept of "pseudo-model" than [15]. We also propose cut-free sequent calculi for our logics, based on the idea introduced in [19] and prove Craig interpolation theorem by Maehara's method [16, 21]. Also, we establish decidability of the sequent calculi by the standard argument [8, 9] on a cut-free derivation of a sequent, while [15] does not show it for their Hilbert systems.

The paper is organized as follows. In Section 2, we introduce syntax and semantics for intuitionistic epistemic logic with distributed knowledge. Section 3 defines Hilbert systems of the logics, and state soundness results. In Section 4, strong completeness of the Hilbert systems of the logics is shown, via a notion of "pseudo-model".

In Section 5, we introduce sequent calculi for the logics and prove the cut-elimination theorem, Craig interpolation theorem, and decidability.

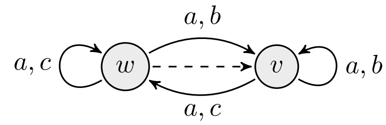

Fig. 1. Example of a frame

2 Syntax and Semantics of Intuitionistic Epistemic Logics with Distributed Knowledge Operators

We denote a finite set of agents by Agt. We call a *nonempty* subset of Agt "group" and denote it by G, H, etc. We denote by Grp the set of all groups, i.e., the set $\wp(\mathsf{Agt}) \setminus \{\varnothing\}$ of all non-empty subset of Agt. Let Prop be a countable set of propositional variables and Form be the set of formulas defined inductively by the following clauses:

$$\mathsf{Form} \ni \varphi ::= p \in \mathsf{Prop} \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid D_G\varphi.$$

We read $D_G\varphi$ as "φ is distributed knowledge among a group G". We define $\neg\varphi$ as $\varphi \to \bot$ and the epistemic operator $K_a\varphi$ (read "agent a knows that φ") as $D_{\{a\}}\varphi$. As noted above, an expression of the form $D_\varnothing\varphi$ is *not* a well-formed formula, since we have excluded $\varnothing$ from our definition of groups.

We introduce Kripke semantics for intuitionistic multi-agent epistemic logic with distributed knowledge, along the lines of [15].

Definition 2.1 (Frame, Model)**.** A tuple $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$ is a *frame* if: W is a set of states; $\leqslant$ is a preorder on W; $(R_a)_{a\in\mathsf{Agt}}$ is a family of binary relations on W, indexed by agents; and $\leqslant; R_a \subseteq R_a$ (for all $a \in \mathsf{Agt}$), where $R_1; R_2 := \{(x, z) \mid$ there exists y such that xR_1y and $yR_2z\}$.

A pair $M = (F, V)$ is a *model* if F is a frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$. We denote an underlying set of states of a frame F or a model M by $|F|$ or $|M|$.

For a model $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model*.

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Satisfaction relation $M, w \Vdash \varphi$ on pointed models and formulas is defined recursively as follows:

$$
\begin{array}{lll}
M, w \Vdash p & \text{iff} & w \in V(p), \\
M, w \Vdash \bot & & \text{Never}, \\
M, w \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \in W, \\
& & \text{if } w \leqslant v \text{ then } M, v \nVdash \varphi \\
& & \text{ or } M, v \Vdash \psi, \\
M, w \Vdash \varphi \wedge \psi & \text{iff} & M, w \Vdash \varphi \text{ and } M, w \Vdash \psi, \\
M, w \Vdash \varphi \vee \psi & \text{iff} & M, w \Vdash \varphi \text{ or } M, w \Vdash \psi, \\
M, w \Vdash D_G\varphi & \text{iff} & \text{for all } v \in W, \\
& & \text{if } (w, v) \in \bigcap_{a \in G} R_a \\
& & \text{ then } M, v \Vdash \varphi.
\end{array}
$$

It is noted from our definition of $K_a\varphi := D_{\{a\}}\varphi$ that the satisfaction of $K_a\varphi$ at a state w of a model M is given as follows:

$$
\begin{array}{ll}
& M, w \Vdash K_a\varphi, \\
\text{iff} & \text{for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \Vdash \varphi.
\end{array}
$$

As is the case with ordinary intuitionistic logic, we have the following heredity property for a formula.

Proposition 2.2 (Heredity). *If $M, w \Vdash \varphi$ and $w \leqslant v$, then $M, v \Vdash \varphi$.*

Proof. By induction on φ. For the case where $\varphi \equiv D_G\psi$, it is noted that the condition $\leqslant ; R_a \subseteq R_a$ of a frame implies that $\leqslant ; \bigcap_{a \in G} R_a \subseteq \bigcap_{a \in G} R_a$. □

Fig. 1 is an example of a frame. The preorder is depicted by a dotted arrow. Note that we omit reflexive arrows for the preorder. If a valuation is defined by, for example, $V(p) = \{v\}$ for any $p \in \mathsf{Prop}$, V satisfies the heredity condition. In this model, it can be seen that different groups have different distributed knowledge even at the same state. Indeed, $D_{\{a,b\}}p$ is true at w, but $D_{\{a,c\}}p$ is false at w. Further, we can also see that seriality for each agent's relation is not always preserved under taking intersection among a group. Namely, R_b and R_c are serial but $R_b \cap R_c$ is not in the example. This is why we should restrict (D) axiom to $\neg D_{\{a\}}\bot$, as defined in Table 1. Given a frame $F = (W, \leqslant, (R_a)_{a \in \mathsf{Agt}})$, we say that a formula φ is *valid* in F (notation: $F \Vdash \varphi$) if $(F, V), w \Vdash \varphi$ for every valuation function V and every $w \in W$. Moreover, a formula φ is valid in a class $\mathbb{F}$ of frames (notation: $\mathbb{F} \Vdash \varphi$) if $F \Vdash \varphi$ for every $F \in \mathbb{F}$.

Definition 2.3. A formula φ is a *semantic consequence* of Γ in a frame class $\mathbb{F}$ if for all frame $F \in \mathbb{F}$, a valuation V on F, a state $w \in |F|$, if $(F, V), w \Vdash \Gamma$, then $(F, V), w \Vdash \varphi$. We write it as "$\Gamma \Vdash_{\mathbb{F}} \varphi$".

3 Hilbert Systems

Hilbert systems for intuitionistic epistemic logics with D_G operators are constructed from axioms and rules shown in Table 1.

Table 1. Axioms and Rules for Hilbert-style Axiomatizations

Axioms and Rules for Intuitionistic Logic	
$(\mathbf{k})$	$\varphi \to (\psi \to \varphi)$
$(\mathbf{s})$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$(\vee\mathbf{i}_1)$	$\varphi \to (\varphi \vee \psi)$
$(\vee\mathbf{i}_2)$	$\psi \to (\varphi \vee \psi)$
$(\vee\mathbf{e})$	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
$(\wedge\mathbf{e}_1)$	$(\varphi \wedge \psi) \to \varphi$
$(\wedge\mathbf{e}_2)$	$(\varphi \wedge \psi) \to \psi$
$(\wedge\mathbf{i})$	$\varphi \to (\psi \to (\varphi \wedge \psi))$
$(\bot)$	$\bot \to \varphi$
$(\mathbf{MP})$	From φ and $\varphi \to \psi$, infer ψ
Axioms and Rules for H($\mathbf{IK}$)	
(Incl)	$D_G\varphi \to D_H\varphi \quad (G \subseteq H)$
(K)	$D_G(\varphi \to \psi) \to (D_G\varphi \to D_G\psi)$
(Nec)	From φ, infer $D_G\varphi$
Additional Axioms for D_G operators	
(T)	$D_G\varphi \to \varphi$
(D)	$\neg D_{\{a\}}\bot$
(4)	$D_G\varphi \to D_G D_G\varphi$

A Hilbert system H($\mathbf{IK}$) consists of axioms and rules for intuitionistic logic, axioms (Incl) and (K), and a rule (Nec). Hilbert systems H($\mathbf{IKT}$), H($\mathbf{IKD}$), H($\mathbf{IK}4$), H($\mathbf{IK}4\mathbf{D}$), and H($\mathbf{IS}4$) are defined as axiomatic expansions of H($\mathbf{IK}$) with (T), (D), (4), (4) and (D), and (T) and (4), respectively. Let $\mathbf{X}$ be any of $\mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IKD}$, $\mathbf{IK}4$, $\mathbf{IK}4\mathbf{D}$, and $\mathbf{IS}4$ in what follows. The notion of provability in each system is defined as usual, and the fact that a formula φ is provable in H($\mathbf{X}$) is denoted by "$\vdash_{\mathsf{H}(\mathbf{X})} \varphi$". We also define derivability relation between a set Γ of formulas and a formula φ as below.

Definition 3.1. A formula φ is *derivable* from Γ in a logic $\mathbf{X}$ if $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \to \varphi$ for some finite set Γ' which is a subset of Γ. We write it as "$\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$".

We introduce a class of frames corresponding to each logic, in order to state soundness of our axiomatization.

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Definition 3.2. A class of frames $\mathbb{F}(\mathbf{X})$ is defined as follows:

— $\mathbb{F}(\mathbf{IK})$ is the class of all frames.

— $\mathbb{F}(\mathbf{IKT})$ is the class of all frames such that R_a is reflexive ($a \in \mathsf{Agt}$).

— $\mathbb{F}(\mathbf{IKD})$ is the class of all frames such that R_a is serial ($a \in \mathsf{Agt}$).

— $\mathbb{F}(\mathbf{IK}4)$ is the class of all frames such that R_a is transitive ($a \in \mathsf{Agt}$).

— $\mathbb{F}(\mathbf{IK}4\mathbf{D})$ is the class of all frames such that R_a is transitive and serial ($a \in \mathsf{Agt}$).

— $\mathbb{F}(\mathbf{IS}4)$ is the class of all frames such that R_a is reflexive and transitive ($a \in \mathsf{Agt}$).

Here, reflexivity, seriality, and transitivity are defined ordinarily.

We can prove the following soundness theorem by induction on φ. Note that axioms (T) and (4) are valid in reflexive and transitive frames, respectively, because if R_a is reflexive or transitive for any $a \in G$, $\bigcap_{a\in G} R_a$ is also reflexive or transitive, respectively.

Theorem 3.3. *If* $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$*, then* $\mathbb{F}(\mathbf{X}) \Vdash \varphi$*.*

4 Completeness

In the present section, we explain a proof of the strong completeness theorem of our logic. Let Γ be a set of formulas and φ be a formula. The strong completeness theorem is stated as follows.

Theorem 4.1. *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$*. Then, if* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X})} \varphi$*, then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$*.*

As in [5], we show the theorem in two steps via the notion of "pseudo-model", that is, we first construct a canonical pseudo-model satisfying truth lemma, and then transform it into an equivalent pseudo-model which can be regarded as a model in the sense of Definition 2.1.

Definition 4.2 (Pseudo-frame, Pseudo-model)**.** A tuple $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$ is a *pseudo-frame* if: $\leqslant; R_G \subseteq R_G$ for any $G \in \mathsf{Grp}$ and $R_H \subseteq R_G$ if $G \subseteq H$.

A pair $M = (F, V)$ is a *pseudo-model* if F is a pseudo-frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$.

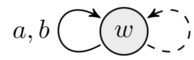

Fig. 2. Example of a pseudo-frame

Example 4.3. Fig. 2 is an example of a pseudo-frame. We name it F_{ex}. Note that $\{a\}$ is written as "a" and $R_{\{a,b\}}$ is defined as $\emptyset$ here. Since $R_{\{a,b\}} = \emptyset$, the condition of "$R_H \subseteq R_G$ if $G \subseteq H$" is self-evidently satisfied, i.e., $R_{\{a,b\}} \subseteq R_{\{a\}}$ and $R_{\{a,b\}} \subseteq R_{\{b\}}$. Note that $R_{\{a\}} \cap R_{\{b\}} \not\subseteq R_{\{a,b\}}$ in F_{ex}, while the contrary is guaranteed by the condition of "$R_H \subseteq R_G$ if $G \subseteq H$". Any frame can be regarded as a pseudo-frame with only relations for singleton groups, as in F_{ex}.

Definition 4.4 (Pseudo-satisfaction Relation)**.** For a pseudo-model M, a state $w \in |M|$, and a formula φ, a *pseudo-satisfaction relation* $M, w \Vdash^{ps} \varphi$ is defined the same as the satisfaction relation $\Vdash$, except for the clause for $D_G\varphi$: that is:

$$\begin{array}{ll} & M, w \Vdash^{ps} D_G\varphi, \\ \text{iff} & \text{for all } v \in W, \text{ if } (w, v) \in R_G \text{ then } M, v \Vdash^{ps} \varphi. \end{array}$$

Namely, in a pseudo-model, an operator D_G is treated like a primitive box operator, parameterized by a group.

Considering the definition of satisfaction relation for $D_G\varphi$, a pseudo-frame can be seen as a frame if the condition $R_G = \bigcap_{a\in G} R_{\{a\}}$ is satisfied for any group G.

So, we can prove the strong completeness theorem by transforming a canonical pseudo-model into a pseudo-model enjoying the condition above without changing satisfaction. We do this by a method of "tree unraveling".

4.1 Canonical Pseudo-Model

We define a canonical pseudo-model of our logics and state some properties of it in the present subsection. Since D_G operators are interpreted as primitive box-like operators indexed by a group in a pseudo-model, a canonical pseudo-model defined here is essentially the same as the canonical model of intuitionistic epistemic logics without distributed knowledge, which is described in detail e.g., in [17, Chapter 1]. Let $\mathbf{X}$ be any of $\mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IKD}$, $\mathbf{IK}4$, $\mathbf{IK}4\mathbf{D}$, and $\mathbf{IS}4$ below.

Definition 4.5 (consistency)**.** A set Γ of formulas is $\mathbf{X}$-*consistent* if $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$.

Definition 4.6 (prime theory)**.** Γ is an $\mathbf{X}$-*prime* theory if:

1. Γ is prime, i.e., if $\varphi_1 \vee \varphi_2 \in \Gamma$, then $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$.
2. Γ is a $\mathbf{X}$-theory, i.e., if $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$, then $\varphi \in \Gamma$.

The following are useful properties of a consistent and prime theory.

Lemma 4.7. *Let a set Γ of formulas be an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory:*

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ *iff* $\varphi \in \Gamma$.
2. *If* $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$, *then* $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \wedge \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
5. $\varphi \vee \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
6. *If* $\varphi \to \psi \notin \Gamma$, *then* $\Gamma \cup \{\varphi\} \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.
7. *If* $D_G\psi \notin \Gamma$, *then* $D_G^{-1}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.

Lemma 4.8 (Lindenbaum)**.** *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. If $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$, then there is an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory Γ^+ such that $\Gamma \subseteq \Gamma^+$ and $\Gamma^+ \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$.*

Definition 4.9. Given a set Γ of formulas, we define $D_G^{-1}\Gamma := \{\varphi \in \mathsf{Form} \mid D_G\varphi \in \Gamma\}$. A *canonical pseudo-model:*

$$M^{\mathbf{X}} = (W^{\mathbf{X}}, \leqslant^{\mathbf{X}}, (R_G^{\mathbf{X}})_{G \in \mathsf{Grp}}, V^{\mathbf{X}}),$$

is defined as follows:

— $W^{\mathbf{X}} :=$ $\{\Gamma \mid \Gamma$ is an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory$\}$.

— $\Gamma \leqslant^{\mathbf{X}} \Delta$ iff $\Gamma \subseteq \Delta$.

— $\Gamma R_G^{\mathbf{X}} \Delta$ iff $D_G^{-1}\Gamma \subseteq \Delta$.

— $V^{\mathbf{X}}(p) := \{\Gamma \in W^{\mathbf{X}} \mid p \in \Gamma\}$.

The definition is well-defined:

Proposition 4.10. $M^{\mathbf{X}}$ is *a pseudo-model.*

Lemma 4.11 (Truth Lemma)**.** *Let Γ be an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory. Then, $\varphi \in \Gamma$ if and only if $M^{\mathbf{X}}, \Gamma \Vdash^{ps} \varphi$.*

Proof. By induction on φ. We show the case $\varphi \equiv D_G\psi$. First, we show the left-to-right. Assume $D_G\psi \in \Gamma$ and fix any $\Delta \in W^{\mathbf{X}}$ such that $\Gamma R_G^{\mathbf{X}} \Delta$, i.e., $D_G^{-1}\Gamma \subseteq \Delta$. Clearly, $\psi \in \Delta$, and by the induction hypothesis, we have $M^{\mathbf{X}}, \Delta \Vdash \psi$. Next, We show the contraposition of the right-to-left. Assume $D_G\psi \notin \Gamma$. By item 7 of Lemma 4.7, and Lemma 4.8, there is an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory Δ such that $D_G^{-1}\Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathsf{H}(\mathbf{X})} \psi$. By item 1 of Lemma 4.7 and induction hypothesis, we have $M^{\mathbf{X}}, \Delta \nVdash \psi$, which shows $M^{\mathbf{X}}, \Gamma \nVdash D_G\psi$. ☐

For each axiom, the canonical pseudo-model satisfies the corresponding property on relations for D_G.

Proposition 4.12. *1. If $\mathbf{X}$ has the axiom (T), $R_G^{\mathbf{X}}$ is reflexive in $M^{\mathbf{X}}$.*

2. If $\mathbf{X}$ has the axiom (D), $R_{\{a\}}^{\mathbf{X}}$ is serial in $M^{\mathbf{X}}$.

3. If $\mathbf{X}$ has the axiom (4), $R_G^{\mathbf{X}}$ is transitive in $M^{\mathbf{X}}$.

Proof. We only show item 2. Fix any $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory Γ. The aim is to find an $\mathbf{X}$-consistent and $\mathbf{X}$-prime theory Δ such that $D_{\{a\}}^{-1}\Gamma \subseteq \Delta$. By Lemma 4.8, it suffices to show $D_{\{a\}}^{-1}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$. Assuming the contrary, we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} \varphi_i \to \bot$ for some $\varphi_i \in D_{\{a\}}^{-1}\Gamma$. By (Nec), (K), and intuitionistic propositional tautologies, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{\{a\}}\varphi_i \to D_{\{a\}}\bot$. Since $D_{\{a\}}\varphi_i \in \Gamma$, it means $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} D_{\{a\}}\bot$. However, we also have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \neg D_{\{a\}}\bot$ by the assumption, which leads to contradiction by item 1 to 3 of Lemma 4.7. ☐

4.2 Tree Unraveling

We introduce a method called “tree unraveling”, which transforms a pseudo-model into another pseudo-model satisfying $\bigcap_{a \in G} R_{\{a\}} = R_G$ (i.e., a model in the sense of Definition 2.1). Our definitions below are intuitionistic generalizations of definitions proposed in [5] over classical logic.

Definition 4.13. Let $M = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}}, V)$ be a pseudo-model. A pseudo-model $M' = (W', \leqslant \cap (W' \times W'), (R_G \cap (W' \times W'))_{G \in \mathsf{Grp}}, V')$ is a *generated submodel* of M if: $W' \subseteq W$; If $w \in W'$ and $w \leqslant w'$ then $w' \in W'$; If $w \in W'$ and wR_Gw' then $w' \in W'$; and $V'(p) = V(p) \cap W'$ for any $p \in \mathsf{Prop}$.

For $X \subseteq |M|$, we define M_X as the smallest generated submodel containing X. If $M = M_X$, we say that M is generated by X.

Definition 4.14. Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$:

— We put $w_0 := w$ and define $\mathrm{Finpath}(F, w)$ as $\{\langle w_0, L_1, w_1, L_2, \cdots, L_n, w_n \rangle \mid n \geq 0,$ $L_i \in \{\leqslant, R_G\}_{G \in \mathsf{Grp}}, w_{i-1} L_i w_i$ for all $1 \leq i \leq n\ \}$. We call an element of $\mathrm{Finpath}(F, w)$ “a path (from a state w)” and denote it by $\overrightarrow{u}, \overrightarrow{v}$, etc.

— For $\overrightarrow{u} = \langle w_0, L_1, w_1, L_2, \cdots, L_{n-1}, w_{n-1}, L_n, w_n \rangle \in \mathrm{Finpath}(F, w)$, $\mathsf{tail}(\overrightarrow{u})$ is defined as w_n.

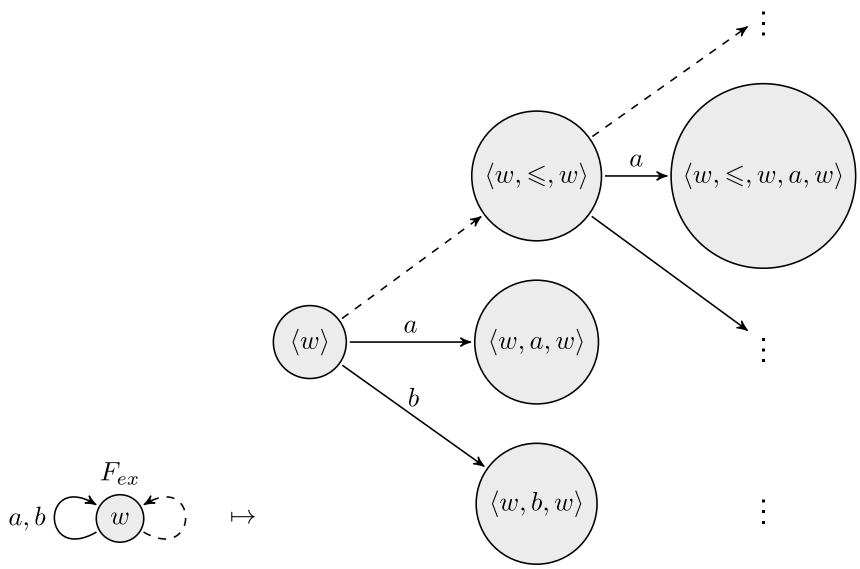

Fig. 3. Tree unraveling

— We say that paths $\overrightarrow{u}, \overrightarrow{v} \in \mathrm{Finpath}(F, w)$ satisfy a relation $\overrightarrow{u} \preccurlyeq \overrightarrow{v}$ if and only if $\overrightarrow{v} \equiv \overrightarrow{u}^\frown\langle \leqslant, w' \rangle$, where $^\frown$ means concatenation of two tuples.

— We say that paths $\overrightarrow{u}, \overrightarrow{v} \in \mathrm{Finpath}(F, w)$ satisfy a relation $\overrightarrow{u}\mathcal{R}_G\overrightarrow{v}$ if and only if $\overrightarrow{v} \equiv \overrightarrow{u}^\frown\langle R_H, w' \rangle$ and $G \subseteq H$.

— A *valuation* $\mathcal{V}$: Prop $\to \mathcal{P}(\mathrm{Finpath}(F, w))$ is defined by:

$$\mathcal{V}(p) = \{\overrightarrow{u} \in \mathrm{Finpath}(W, w) \mid \mathsf{tail}(\overrightarrow{u}) \in V(p)\}.$$

Take F_{ex} in Fig. 2 as an example. The set $\mathrm{Finpath}(F_{ex}, w)$ of paths on F_{ex} and $\preccurlyeq$ and $\mathcal{R}_G$ on this set are drawn in Fig. 3. The point is that the a-arrow and b-arrow on w in F_{ex} are transformed into two arrows with different destinations, so that the condition "$R_{\{a\}} \cap R_{\{b\}} = R_{\{a,b\}}$" is not satisfied in F_{ex} but becomes satisfied in $\mathrm{Finpath}(F_{ex}, w)$. However, as it is, $(\mathrm{Finpath}(F_{ex}, w), \preccurlyeq, (\mathcal{R}_G)_{G\in\mathsf{Grp}})$ is not a pseudo-frame, since $\preccurlyeq$ itself is not a preorder and the condition "$\leqslant ; R_G \subseteq R_G$" is not satisfied because, for example, there is no a-arrow from $\langle w \rangle$ to $\langle w, \leqslant, w, a, w \rangle$. Therefore, a preorder and relations for D_G on $\mathrm{Finpath}(F, w)$ in general should be defined as follows.

Definition 4.15 (Tree Unraveling)**.** Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$. A *tree unraveling pseudo-model* $\mathrm{Tree}(M, w)$ of a pointed pseudo-model (M, w) is defined as a tuple:

$$(\mathrm{Finpath}(F, w), \preccurlyeq^*, (\preccurlyeq^*; \mathcal{R}_G)_{G\in\mathsf{Grp}}, \mathcal{V}),$$

where R^* is defined as the reflexive and transitive closure of a relation R.

We can easily show that $\mathrm{Tree}(M, w)$ is indeed a pseudo-model. Moreover, as explained above with Fig. 3, $\bigcap_{a\in G} \mathcal{R}_{\{a\}} = \mathcal{R}_G$ holds, from which it is also shown that $\bigcap_{a\in G} \preccurlyeq^*; \mathcal{R}_{\{a\}} = \preccurlyeq^*; \mathcal{R}_G$ by a simple argument using property of a tree unraveling pseudo-model. Therefore, $\mathrm{Tree}(M, w)$ can be seen as a model in the sense of Definition 2.1. The following is a key property of tree unraveling.

Lemma 4.16. *Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$. Then, $M, w \Vdash^{ps} \varphi$ iff $\mathrm{Tree}(M, w), \langle w \rangle \Vdash^{ps} \varphi$ for any formula φ.*

Proof. The function $\overrightarrow{u} \mapsto \mathsf{tail}(\overrightarrow{u})$ is a bounded morphism (which takes not only relations for D_G but also a preorder into account) from $\mathrm{Tree}(M, w)$ to M. $\square$

We end the present section by proving Theorem 4.1.

Proof. (Outline) First, we show the case of $\mathbf{IK}$. We show the contraposition. Assume $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$. By Lemma 4.8, We can find an $\mathbf{X}$-prime and $\mathbf{X}$-consistent theory Γ^+ such that $\Gamma \subseteq \Gamma^+$ and $\Gamma^+ \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$. Since $\Gamma \subseteq \Gamma^+$, $M^{\mathbf{X}}, \Gamma^+ \Vdash^{ps} \Gamma$ by the left-to-right of Lemma 4.11. On the other hand, $M^{\mathbf{X}}, \Gamma^+ \nVdash^{ps} \varphi$ by the right-to-left of Lemma 4.11 and item 1 of Lemma 4.7. We can take $\mathrm{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$, because, by Proposition 4.10, $M^{\mathbf{X}}_{\Gamma^+}$ is a pseudo-model generated by Γ^+. Since any tree unraveling pseudo-model can be seen as a model in the sense of Definition 2.1, it suffices to show that $(M^{\mathbf{X}}, \Gamma^+)$ satisfies exactly the same formulas as $\left(\mathrm{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right), \langle \Gamma^+ \rangle\right)$. First, $(M^{\mathbf{X}}, \Gamma^+)$ satisfies exactly the same formulas as $(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+)$. Then, by Lemma 4.16, $(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+)$ satisfies exactly the same formulas as $\left(\mathrm{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right), \langle \Gamma^+ \rangle\right)$.

For the remaining logics, basically, a similar argument can be applied, but definitions and proofs become more involved. In order to make relations for D_G have the desired property, such as reflexivity or transitivity, the relation $\preccurlyeq^*; \mathcal{R}_G$ should be replaced by $\preccurlyeq^*; \mathcal{R}^\circ_G$, $(\preccurlyeq^*; \mathcal{R}^+_G)^+$, and $(\preccurlyeq^*; \mathcal{R}^*_G)^*$ for $\mathbf{IKT}$, $\mathbf{IK}4$ and $\mathbf{IK}4\mathbf{D}$, and $\mathbf{IS}4$, respectively, in the definition of tree unraveling. Here, R° and R^+ are defined as the reflexive closure and transitive closure of a relation R, respectively. Also, note that $\preccurlyeq^*; \mathcal{R}_G$ and $\left(\preccurlyeq^*; \mathcal{R}^+_G\right)^+$ are serial if R_G is serial and that $R^{\mathbf{X}}_{\{a\}}$ is serial if $\mathbf{X}$ has the axiom (D) (by Proposition 4.12). The resulting tree unravelings are also easily shown to be pseudo-models. The condition "$\bigcap_{a \in G} R_{\{a\}} = R_G$" in the tree unraveling pseudo-models also can be shown to be satisfied, by using the property of a tree unraveling pseudo-model. Therefore, from the above argument, $\mathrm{Tree}\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$ can be seen as a model, whose underlying frame is an element of $\mathbb{F}(\mathbf{X})$. The fact that the function $\overrightarrow{u} \mapsto \mathsf{tail}(\overrightarrow{u})$ is a bounded morphism also in the respective tree unraveling pseudo-models is needed, and can be shown straightforwardly. $\square$

5 Sequent Calculi of Intuitionistic Epistemic Logics with Distributed Knowledge

5.1 Equipollence and Cut-Elimination

A *sequent* is a pair of finite multisets of formulas Γ and Δ denoted by "$\Gamma \Rightarrow \Delta$", where $\#\Delta \leq 1$. The multiset Γ is called an"antecedent" of a sequent $\Gamma \Rightarrow \Delta$, and Δ a "succedent". A sequent is intuitively interpreted as "if all formulas in Γ hold, then a formula in Δ holds." The reason why the number of Δ is restricted is that we build our calculus on the basis of Gentzen's **LJ** [8, 9] for intuitionistic propositional logic. Our sequent calculi for the intuitionistic epistemic logics with distributed knowledge are presented in Table 2. Axioms, structural rules, and propositional logical rules are common to $\mathbf{LJ}$. The other rules are the same as the ones in [19], except that rules for (D) axiom, i.e., $(D_{\mathbf{IKD}})$ and $(D_{\mathbf{IK4D}})$ are added, in order to construct calculi for the logics $\mathbf{IKD}$ and $\mathbf{IK}4\mathbf{D}$.

We note that when $n = 0$, e.g., in the rule (D) of Table 2, the multiset is regarded as the empty multiset and thus $\bigcup_{i=1}^n G_i$ is regarded as $\varnothing$. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in each calculus $\mathsf{G}(\mathbf{X})$ if there exists a finite tree of sequents, whose root is $\Gamma \Rightarrow \Delta$ and each node of which is inferred by some rule (including axioms) in $\mathsf{G}(\mathbf{X})$. We write it as $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$.

Example 5.1. The following is an application of rule (D), which captures typical inference involving distributed knowledge mentioned in Introduction:

$$\frac{p \to q, p \Rightarrow q}{D_{\{a\}}(p \to q), D_{\{b\}}p \Rightarrow D_{\{a,b\}}q}\ (D).$$

We note that for any logic $\mathbf{X}$ under consideration, $\mathsf{H}(\mathbf{X})$ and $\mathsf{G}(\mathbf{X})$ are equipollent in the following sense.

Theorem 5.2 (Equipollence)**.** *Let $\mathbf{X}$ be any of $\mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IKD}$, $\mathbf{IK}4$, $\mathbf{IK}4\mathbf{D}$, and $\mathbf{IS}4$. Then, the following hold. 1. If $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, then $\vdash_{\mathsf{G}(\mathbf{X})} \Rightarrow \varphi$. 2. If $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma \to \bigvee \Delta$, where $\bigwedge \varnothing := \top$ and $\bigvee \varnothing := \bot$.*

Proof. We show the case of $\mathbf{IK}$. The idea for proof is common to the rest. Here we focus on item 2 alone. We show item 2 by induction on the structure of the derivation for the sequent $\Gamma \Rightarrow \Delta$. We deal with the case for the rule (D) only. Suppose we have a derivation:

$$\frac{\dfrac{\mathcal{D}}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad \left(\bigcup_{i=1}^n G_i \subseteq G\right)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

We show $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_G\psi$. We have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n \varphi_i \to \psi$ as the induction hypothesis for the derivation $\mathcal{D}$. From this, we can infer by necessitation $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^n \varphi_i \to \psi)$. By this and axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} D_G(\bigwedge_{i=1}^n \varphi_i) \to D_G\psi$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_G\varphi_i \to D_G\psi$. Therefore, it suffices to show that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to \bigwedge_{i=1}^n D_G\varphi_i$, which is equivalent to $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_G\varphi_i$ for any $i \in \{1, \ldots, n\}$. This is evident because we have a theorem in intuitionistic propositional logic $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^n D_{G_i}\varphi_i \to D_{G_i}\varphi_i$ and the axiom (Incl) $\vdash_{\mathsf{H}(\mathbf{X})} D_{G_i}\varphi_i \to D_G\varphi_i$. $\square$

Table 2. Sequent Calculi for $\mathbf{IK}$, $\mathbf{IKT}$, $\mathbf{IKD}$, $\mathbf{IK}4$, $\mathbf{IK}4\mathbf{D}$, and $\mathbf{IS}4$

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (Id) \qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to) \qquad \frac{\Gamma_1 \Rightarrow \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^1) \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow^2)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee^2)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

Logical Rules for D_G of $\mathbf{IK}$

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D)$$

Logical Rules for D_G of $\mathbf{IKT}$

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

Logical Rules for D_G of $\mathbf{IKD}$

$$\frac{\varphi_1, \ldots, \varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D) \qquad \frac{\Gamma \Rightarrow}{D_{\{a\}}\Gamma \Rightarrow}\ (D_{\mathbf{IKD}})$$

Logical Rules for D_G of $\mathbf{IK}4$

$$\frac{\varphi_1, \ldots, \varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IK}4})$$

Logical Rules for D_G of $\mathbf{IK}4\mathbf{D}$

$$\frac{\varphi_1, \ldots, \varphi_n, D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IK}4})$$

$$\frac{\Gamma, D_{\{a\}}\Gamma \Rightarrow}{D_{\{a\}}\Gamma \Rightarrow}\ (\Rightarrow D_{\mathbf{IK}4\mathbf{D}})$$

Logical Rules for D_G of $\mathbf{IS}4$

$$\frac{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow \psi \quad (\bigcup_{i=1}^n G_i \subseteq G)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (\Rightarrow D_{\mathbf{IS}4})$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{D_G\varphi, \Gamma \Rightarrow \Delta}\ (D \Rightarrow)$$

We have the cut-elimination theorem for all of the logics in consideration.

Theorem 5.3 (Cut-Elimination)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$*. Then, the following holds: If* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$*, where* $\mathsf{G}^-(\mathbf{X})$ *denotes a system "*$\mathsf{G}(\mathbf{X})$ *minus the cut rule".*

Proof. First, we introduce a notion of "principal formula". A principal formula is defined for each inference rule, except for the axioms and (Cut) rule and is informally expressed as "a formula, on which the inference rule acts".

Definition 5.4. A *principal formula* of the structural rules, the propositional logical rules, and the rule $(D \Rightarrow)$ is a formula appearing in the lower sequent, which is not contained in Γ_1, Γ_2, Γ, or Δ. A *principal formula* of the rules for D_G operator other than $(D \Rightarrow)$ is every formula in the lower sequent.

To prove the theorem, we consider a system $\mathsf{G}^*(\mathbf{X})$, in which the cut rule is replaced by a "extended" cut rule defined as:

$$\frac{\Gamma \Rightarrow \varphi^n \quad \varphi^m, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Theta}\ (ECut),$$

where φ^n denotes the multi-set of n-copies of φ and $n = 0, 1$ and $m \geq 0$. Since $(ECut)$ is the same as (Cut) when we set $n = m = 1$, it is obvious that if $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, so it suffices to show that if $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{G}^-(\mathbf{X})} \Gamma \Rightarrow \Delta$.

Suppose $\vdash_{\mathsf{G}^*(\mathbf{X})} \Gamma \Rightarrow \Delta$ and fix one derivation for the sequent. To obtain an $(ECut)$-free derivation of $\Gamma \Rightarrow \Delta$, it is enough to concentrate on a derivation whose root is derived by $(ECut)$ and which has no other application of $(ECut)$. In what follows, we let $\mathbf{X}$ be $\mathbf{IK}$. Let us suppose that $\mathcal{D}$ has the following structure:

$$\frac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow \varphi^n}\ (\mathbf{rule}_{\mathcal{L}}) \quad \dfrac{\mathcal{R}}{\varphi^m, \Sigma \Rightarrow \Theta}\ (\mathbf{rule}_{\mathcal{R}})}{\Gamma, \Sigma \Rightarrow \Theta}\ (ECut),$$

where the derivations $\mathcal{L}$ and $\mathcal{R}$ has no application of $(ECut)$ and $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are meta-variables for the name of rule applied there. Let the number of logical symbols (including D_G) appearing in φ be $c(\mathcal{D})$, and the number of sequents in $\mathcal{L}$ and $\mathcal{R}$ be $w(\mathcal{D})$. We show the lemma by double induction on $(c(\mathcal{D}), w(\mathcal{D}))$. If $n = 0$ or $m = 0$, we can derive the root sequent of $\mathcal{D}$ without using $(ECut)$ by weakening rules. So, in what follows we assume $n = 1$ and $m > 0$. Then, it is sufficient to consider the following four cases: [1]

1. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is an axiom.
2. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a structural rule.
3. $\mathbf{rule}_{\mathcal{L}}$ or $\mathbf{rule}_{\mathcal{R}}$ is a logical rule and a cut formula φ is not principal (in the sense we have specified above) for that rule.
4. $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are both logical rules (including (D)) for the same logical symbol and a cut formula φ is principal for each rule.

We concentrate on a rule (D) and the case involving the rule (D) is case 4 only, so we only comment on case 4 where both $\mathbf{rule}_{\mathcal{L}}$ and $\mathbf{rule}_{\mathcal{R}}$ are rules (D). In that case, the given derivation $\mathcal{D}$ has the following structure:

$$\dfrac{\dfrac{\mathcal{L}}{\Gamma \Rightarrow D_G\psi} \quad \dfrac{\mathcal{R}}{(D_G\psi)^m, \Sigma \Rightarrow D_H\chi}}{\Gamma, \Sigma \Rightarrow D_H\chi}\ (ECut),$$

where

$$\mathcal{L} \equiv \dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^n G_i \subseteq G),}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D)$$

and

$$\mathcal{R} \equiv \dfrac{\dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m \Rightarrow \chi} \quad (G \cup \bigcup_{j=1}^m H_j \subseteq H)}{(D_G\psi)^m, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (D).$$

The derivation $\mathcal{D}$ can be transformed into the following derivation $\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\mathcal{L}'}{\varphi_1, \ldots, \varphi_n \Rightarrow \psi} \quad \dfrac{\mathcal{R}'}{\psi^m, \psi_1, \ldots, \psi_m \Rightarrow \chi}}{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \Rightarrow \chi}\ (ECut)}{D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi}\ (D),$$

where the rule (D) is applicable because we have $\bigcup_{i=1}^n G_i \cup \bigcup_{j=1}^m H_j \subseteq H$ by $\bigcup_{i=1}^n G_i \subseteq G$ and $G \cup \bigcup_{j=1}^m H_j \subseteq H$. We call $\mathcal{E}'$ its subderivation whose root sequent is $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \Rightarrow \chi$. The derivation $\mathcal{E}'$ have no application of $(ECut)$ and $c(\mathcal{E}') < c(\mathcal{D})$. Hence, by induction hypothesis, there exists an $(ECut)$-free derivation $\tilde{\mathcal{E}'}$ having the same root sequent. Replacing the derivation $\mathcal{E}'$ by $\tilde{\mathcal{E}'}$ in $\mathcal{E}$, we obtain an $(ECut)$-free derivation for the sequent $D_{G_1}\varphi_1, \ldots, D_{G_n}\varphi_n, D_{H_1}\psi_1, \ldots, D_{H_m}\psi_m \Rightarrow D_H\chi$ as required. □

[1] In case 4, we assume the condition for both rule applications, because if the one of the two rule applications does not satisfy the condition, the whole derivation should be categorized into one of the rest cases.

The following subformula property is an important corollary of the cut-elimination theorem, and later used in a proof of decidability.

Corollary 5.5 (Subformula Property)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$ *and suppose* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$*. Then, there exists a derivation of* $\Gamma \Rightarrow \Delta$ *satisfying a condition that any formula occurring in the derivation is a subformula of certain formula in* Γ *or* Δ*.*

Proof. A cut-free derivation of $\Gamma \Rightarrow \Delta$ satisfies the condition, because any formula in the upper sequent is a subformula of certain formula in the lower sequent in every inference rules of our calculi except (Cut). □

5.2 Craig Interpolation Theorem and Decidability

In many logics, the Craig interpolation theorem can be derived as an application of the cut-elimination theorem, using a Maehara method originally described in [16]. An application of the method to basic modal logic can also be found in [21]. Unlike [19], the concept of 'partition' is simplified, because we do not allow multiple formulas to appear in the succedent of a sequent.

Definition 5.6 (Partition)**.** A *partition* for a sequent $\Gamma \Rightarrow \Delta$ is defined as a tuple $\langle \Gamma_1; \Gamma_2 \rangle$, such that $\Gamma = \Gamma_1, \Gamma_2$.

Definition 5.7. For a formula φ, $\mathsf{Prop}(\varphi)$ is defined as the set of all propositional variables appearing in φ. For a multiset Γ of formulas, $\mathsf{Prop}(\Gamma)$ is defined as $\bigcup_{\varphi \in \Gamma} \mathsf{Prop}(\varphi)$. Similarly, $\mathsf{Agt}(\varphi)$ is defined as the set of agents appearing in φ and $\mathsf{Agt}(\Gamma)$ as $\bigcup_{\varphi \in \Gamma} \mathsf{Agt}(\varphi)$

The following is a key lemma for Craig Interpolation Theorem.

Lemma 5.8. *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$*. Suppose* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$*. Then, for any partition* $\langle \Gamma_1; \Gamma_2 \rangle$ *for the sequent* $\Gamma \Rightarrow \Delta$*, there exists a formula* φ *called "interpolant", satisfying the following:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma_1 \Rightarrow \varphi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi, \Gamma_2 \Rightarrow \Delta$.
2. $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \Delta)$.
3. $\mathsf{Agt}(\varphi) \subseteq \mathsf{Agt}(\Gamma_1) \cap \mathsf{Agt}(\Gamma_2, \Delta)$.

Proof. We prove the case of $\mathbf{IK}$ by induction on the structure of a derivation for $\Gamma \Rightarrow \Delta$. Fix the derivation and name it $\mathcal{D}$. By Theorem 5.3, we can assume that $\mathcal{D}$ is cut-free. We treat only the case of (D) below (for other cases, the reader is referred to [21]). Suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{E}}{\varphi_1,\dots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

A partition of $D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi$ is of the following form:

$$\langle D_{G_1}\varphi_1,\dots,D_{G_k}\varphi_k; D_{G_{k+1}}\varphi_{k+1},\dots,D_{G_n}\varphi_n\rangle.$$

The induction hypothesis on $\mathcal{E}$ for a partition $\langle \varphi_1,\dots,\varphi_k;\varphi_{k+1},\dots,\varphi_n\rangle$ is used. That is, we have derivations for $\varphi_1,\dots,\varphi_k \Rightarrow \chi$ and $\chi,\varphi_{k+1},\dots,\varphi_n \Rightarrow \psi$ for some formula χ. If $k > 0$, we can choose $D_{\bigcup_{i=1}^{k} G_i}\chi$ as a required interpolant, because we have following derivations:

$$\dfrac{\dfrac{\text{I.H.}}{\varphi_1,\dots,\varphi_k \Rightarrow \chi} \quad (\bigcup_{i=1}^{k} G_i \subseteq \bigcup_{i=1}^{k} G_i)}{D_{G_1}\varphi_1,\dots,D_{G_k}\varphi_k \Rightarrow D_{\bigcup_{i=1}^{k} G_i}\chi}\ (D)$$

$$\dfrac{\dfrac{\text{I.H.}}{\chi,\varphi_{k+1},\dots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{k} G_i \cup \bigcup_{i=k+1}^{n} G_i = \bigcup_{i=1}^{n} G_i \subseteq G)}{D_{\bigcup_{i=1}^{k} G_i}\chi, D_{G_{k+1}}\varphi_{k+1},\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

Furthermore, the interpolant enjoys the condition 2 and 3 as induction hypothesis and simple calculation show. If $k = 0$, we can choose χ as an interpolant, since we have the following derivations:

$$\dfrac{\text{I.H.}}{\Rightarrow \chi} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}}{\varphi_1,\dots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D)}{\chi, D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (w\Rightarrow)$$

□

Theorem 5.9 (Craig Interpolation Theorem)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$*. Given that* $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \psi$*, there exists a formula* χ *satisfying the following conditions:*

1. $\vdash_{\mathsf{G}(\mathbf{X})} \varphi \Rightarrow \chi$ *and* $\vdash_{\mathsf{G}(\mathbf{X})} \chi \Rightarrow \psi$*.*
2. $\mathsf{Prop}(\chi) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$*.*
3. $\mathsf{Agt}(\chi) \subseteq \mathsf{Agt}(\varphi) \cap \mathsf{Agt}(\psi)$*.*

We note that not only the condition for propositional variables but also the condition for agents can be satisfied.

Proof. When we set $\Gamma := \varphi$ and $\Delta := \psi$, and take a partition $\langle \Gamma; \varnothing\rangle$, Lemma 5.8 proves Craig Interpolation Theorem. □

Further, decidability of the logics we investigate also follows from the cut-elimination theorem (Theorem 5.3). To show decidability, we introduce a notion of "(1-)reduced sequent".

Definition 5.10. A sequent $\Gamma \Rightarrow \Delta$ is called *reduced* if every formula occurs at most three times in Γ. A sequent $\Gamma \Rightarrow \Delta$ is called *1-reduced* if every formula occurs at most once in Γ.

Definition 5.11. For any sequent $\Gamma \Rightarrow \Delta$, a sequent $\Gamma^* \Rightarrow \Delta$ is a *1-reduced contraction* of $\Gamma \Rightarrow \Delta$ if $\Gamma^* \Rightarrow \Delta$ can be derived from $\Gamma \Rightarrow \Delta$ by applying $(c\Rightarrow)$ to $\Gamma \Rightarrow \Delta$ and is 1-reduced. Clearly, a 1-reduced contraction is determined uniquely.

Proposition 5.12. $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$ *if and only if* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma^* \Rightarrow \Delta$*.*

Proof. By definition of the 1-reduced contraction, the left-to-right is obvious. The right-to-left is also easily shown by applying $(w\Rightarrow)$ to $\Gamma^* \Rightarrow \Delta$. □

Lemma 5.13. *Suppose that* $\vdash_{\mathsf{G}(\mathbf{X})} \Gamma \Rightarrow \Delta$*. Then, there exists a derivation of* $\Gamma^* \Rightarrow \Delta$ *such that the derivation is cut-free and has only reduced sequents.*

Proof. Thanks to Theorem 5.3, we can take a cut-free derivation of $\Gamma \Rightarrow \Delta$. We name it $\mathcal{D}$. We show by induction on the height of $\mathcal{D}$. We treat only the case where the last rule application of $\mathcal{D}$ is (D). That is, suppose $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\mathcal{D}'}{\varphi_1,\dots,\varphi_n \Rightarrow \psi} \quad (\bigcup_{i=1}^{n} G_i \subseteq G)}{D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n \Rightarrow D_G\psi}\ (D).$$

By induction hypothesis, we have a derivation $\mathcal{E}'$ of $(\varphi_1,\dots,\varphi_n)^* \Rightarrow \psi$ such that $\mathcal{E}'$ is cut-free and has only reduced sequents. Applying the rule (D) to $\mathcal{E}'$, we obtain the desired derivation of $(D_{G_1}\varphi_1,\dots,D_{G_n}\varphi_n)^* \Rightarrow D_G\psi$. □

Remark 5.14. We admit three occurrences of the same formula in a reduced sequent, because if we only allow at most two occurrences, induction fails in the case of $(\to\Rightarrow)$ in the proof of this lemma.

Theorem 5.15 (Decidability)**.** *Let* $\mathbf{X}$ *be any of* $\mathbf{IK}$*,* $\mathbf{IKT}$*,* $\mathbf{IKD}$*,* $\mathbf{IK}4$*,* $\mathbf{IK}4\mathbf{D}$*, and* $\mathbf{IS}4$*. A logic* $\mathbf{X}$ *is decidable, that is, there is an algorithm checking whether each sequent* $\Gamma \Rightarrow \Delta$ *has a derivation in* $\mathsf{G}(\mathbf{X})$ *or not.*

Proof. We describe a rough sketch of the proof, based on [21, p. 228]. By Proposition 5.12, it suffices to check whether $\Gamma^* \Rightarrow \Delta$ has a derivation. In what follows, by "tree (of $\Sigma \Rightarrow \Theta$)", we mean a tree of sequents (ending with $\Sigma \Rightarrow \Theta$), whose leaves are axioms, or sequents, to which no rule can be applied. Without any restriction, there are infinitely many trees of $\Gamma^* \Rightarrow \Delta$. Therefore, in order to execute a brute-force search, we impose three restrictions on the trees. In general, if a derivation exists, Lemma 5.13 allows us to find a derivation such that (i) it is cut-free and (ii) it has only reduced sequents. By Corollary 5.5, it has subformula property. Therefore, there are finitely many reduced sequents that can be a part of the derivation. Moreover, we can safely assume that (iii) for each path in the derivation from the root sequent to an initial sequent, each sequent in the path occurs exactly once, because, if there are multiple occurrences of the same sequent, we can always eliminate the redundant occurrences by grafting the subderivation for the uppermost occurrence onto the lowermost occurrence. From the above argument, if we impose the conditions (i) to (iii) on the trees of $\Gamma^* \Rightarrow \Delta$, the number of trees becomes finite and we can construct an algorithm enumerating all of them which also checks whether each tree is a derivation or not. If the algorithm does not find any derivation, we can conclude that $\Gamma^* \Rightarrow \Delta$ has no derivation. $\Box$

6 Concluding Remark

We conclude this paper with four possible directions for further research. The first direction is to simplify our semantic completeness argument via a similar method given in [30] for classical epistemic logic with distributed knowledge. One of the merits of the method is that the notion of pseudo- (or pre-) model is not necessary. The second direction is to add $\mathbf{S}5$-type axioms to our intuitionistic epistemic logic with distributed knowledge. Since Ono [20] showed that there are at least four distinct $\mathbf{S}5$-type axioms over the intuitionistic modal logic $\mathbf{S}4$, it would be interesting to study the corresponding $\mathbf{S}5$-type axioms in our setting. The third direction is to expand our syntax with the common knowledge operator (cf. [29]). This amounts to investigating the intuitionistic counterpart of [30]. The final direction is to consider dynamic expansions of our syntax. In order to formalize changes of agents' distributed knowledge, for example, we may add public announcement operators [22, 4] or resolution operators [1].

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Corresponding author is Ryo Murai.