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On the Algebrization of the Multi-valued Logics CG'_3 and G'_3

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Abstract. Multi-valued logics form a family of formal languages with several applications in computer sciences, particularly in the field of Artificial intelligence. Paraconsistent multi-valued logics have been successful applied in logic programming, fuzzy reasoning, and even in the construction of paraconsistent neural networks. $\mathbf{G}'_3$ is a 3-valued logic with a single represented truth value by 1. $\mathbf{CG}'_3$ is a paraconsistent, 3-valued logic that extends $\mathbf{G}'_3$ with two truth values represented by 1 and 2. The state of the art of $\mathbf{CG}'_3$ comprises a Kripke semantics and a Hilbert axiomatization inspired by the Lindenbaum-Łos technique. In this work, we show that $\mathbf{G}'_3$ and $\mathbf{CG}'_3$ are algebrizable in the sense of Blok and Pigozzi. These results may apply to the development of paraconsistent reasoning systems.

Keywords. Paraconsistent logics, blok-pigozzi algebrization, non-monotonic reasoning.

1 Introduction

In computer science, it is well known the successful application of logics as a foundation of programming languages, that is, programs can be characterized as proofs in logical inference systems (Curry-Howard isomorphism) [11]. In development Artificial Intelligence (AI), logical languages have also played a key role: in the burgeoning of reasoning systems and even as a tool for proving algorithm correctness. There are even cases in which some logical theories have served for the advanced programming paradigms, such as logic programming [25].

Paraconsistent logics form a family of languages designed to analyze and reason from inconsistencies (from the point of view of classical logic), as is often useful in many AI contexts, such as signal and image processing and expert systems [26]. Within the family of paraconsistent logics, Annotated logics, which encompass fuzzy set theory, are the most widely applied in AI [1]. Another scope of paraconsistent logic is non-monotonic reasoning, a fundamental notion in the development of intelligent systems.

In [2, 3], it is introduced a standard semantics for non-monotonic reasoning in the setting of annotated logics and annotated logic programs. Multi-valued logics are non-classical logics [5]. Like in logic classical, multi-valued logics also enjoy the principle of truth functionality: the truth value of a compound sentence is determined through the truth values of its component sentences and remains the same when one of the component sentences is replacing by another sentence with the same truth value. However, in contrast to the classical case, multi-valued logics do not restrict the number of truth values to just two. A larger set of degrees of truth is the distinctive feature in the context of many-valued logics.

In [15], it is reported a detailed summary of multi-valued logics. Some multi-valued logic systems are presented as families of systems of uniformly defined finite and infinite values, for example, Łukasiewicz logic, Gödel's logic, systems based on the t-norm, 3-valued systems, Dunn-Belnap's 4-value system. Most common inference systems for multi-valued logics are Hilbert and Gentzen (sequent) calculus, and Tableaux [15]. A broad class of infinitely valued logics is described by [20].

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Classical logic, as well as intuitionistic logic, suffer a disadvantage when reasoning with inconsistent information. According to the principle of explosion, also known as "*ex contradictione sequitur quodlibet*", all theory or inconsistent knowledge base is trivial. Classical logic is then useless to reason with inconsistencies. As a result, alternatives to classical logic that do not have this drawback have been developed, called "paraconsistent" approaches. In 1954 F. Asenjo, in his doctoral dissertation, proposes for the first time to use multi-valued logic as a form of paraconsistent logic (logics whose logical consequence relationship semantics or proof theory is not explosive [16]). The focus of many truth values is to abandon the classical assumption and allow more than two values. The most common strategy is to use three truth values: true, false, and both (true and false) for evaluation of formulas.

George Boole introduced the algebra of logic or algebraic logic in [7] as an explicit algebraic system showing the underlying mathematical structure of logic. The methodology started by Boole was continued in the 19th century for the work of A. De Morgan, W. S. Jevons, C. S. Peirce, and E. Schröder. A summary of these works can be found in [8]. The relationship between logic and algebra from the contemporary perspective goes back to the ideas of Lindenbaum and Tarski, as follows: formulas of a given logic are interpreted in algebras with operations associated with the logical connectives. In [6], Blok and Pigozzi proposed a generalization of the techniques of original algebra to encompass a broader range of logics. Generalization of the Blok and Pigozzi method was suggested in the literature [12, 13, 14]. Algebraic foundations for logic have been shown useful in the development of reasoning systems [10, 17]. In this paper, we show that $\mathbf{CG}'_3$ and $\mathbf{G}'_3$ are Blok-Pigozzi algebrizable. We believe this result may help in the development of paraconsistent reasoning systems.

This article is organized as follows: in Section 2, we present some known definitions and results according to the setting of the present manuscript; in Section 3, we study the $\mathbf{CG}'_3$ and $\mathbf{G}'_3$ logics which are defined in terms of four connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ where the implication is deductive to $\mathbf{CG}'_3$. The main result of the paper is also described in this Section, that is, it is shown that $\mathbf{CG}'_3$ and $\mathbf{G}'_3$ are algebraizable logics with the Blok-Pigozzi method. Finally, in the last Section, we give a summary of the paper and we describe a list of open problems to be studied in the future.

2 Background

We first introduce the syntax of the logical formulas considered in this paper. We follow standard notation and basic definitions as W. Carnielli and M.Coniglio in [9].

Definition 1 (Propositional signatures)**.** *A proposition signature is a set Θ of symbols called connectives, together with the information concerning the arity of each connective.*

The following symbols will be used for logical connectives: $\wedge$ (conjunction, binary); $\vee$ (disjunction, binary); $\rightarrow$ (implication, binary); $\neg$ (weak negation, unary); $\bullet$ (inconsistency operator, unary); $\sim$ (strong negation, unary); $\perp$ (bottom formula, 0-ary).

Definition 2 (Propositional language)**.** *Let $Var = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables, and let Θ be any propositional signature. The propositional language generated by Θ from Var will be denoted by $\mathcal{L}_\Theta$.*

Definition 3 (Standard logic)**.** *A logic $\mathscr{L}$ defined over a language $\mathcal{L}$ which has a consequence relation $\vdash$, is Tarskian if it satisfies the following properties, for every $\Gamma \cup \Delta \cup \{\alpha\} \subseteq \mathcal{L}$:*

(i) If $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;

(ii) If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$;

(iii) If $\Delta \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Delta$, then $\Gamma \vdash \alpha$.

A logic satisfying item (ii) above is called monotonic. A logic $\mathscr{L}$ is said to be finitary if it satisfies the following:

(iv) If $\Gamma \vdash \alpha$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \alpha$.

A logic $\mathscr{L}$ defined over a propositional language $\mathcal{L}$ generated by a signature from a set of propositional variables is namely structural, if it satisfies the following property:

(v) Is $\Gamma \vdash \alpha$ then $\sigma[\Gamma] \vdash \sigma[\alpha]$, for every substitution σ of formulas for variables.

Propositional logic is standard if it is Tarskian, finitary, and structural.

From now on, a logic $\mathscr{L}$ will be represented by a pair $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ and $\vdash$ denote the language and the consequence relation of $\mathcal{L}$, respectively. $\mathcal{L}$ is generated by a propositional signature Θ from Var, this is, $\mathcal{L} = \mathcal{L}_\Theta$ then we will write $\mathscr{L} = \langle \Theta, \vdash \rangle$.

Let $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic. Let α be a formula in $\mathcal{L}$ and let $X_1 \dots X_n$ be a finite sequence (for $n \geq 1$) such that each X_i is either a set for formulas in $\mathcal{L}$ or formula in $\mathcal{L}$. Then, as usual, $X_1, \dots, X_n \vdash \alpha$ will stand for $X_1' \cup \dots \cup X_n' \vdash \alpha$, where, for each i, X_i' is X_i, if X_i is a set of formulas, or X_i' is $\{X_i\}$ if X_i is a formula.

Definition 4 (Paraconsistent logic)**.** *A Tarskian logic $\mathscr{L}$ is paraconsistent if it has a (primitive or defined) negation $\neg$ such that $\alpha, \neg\alpha \nvdash_{\mathscr{L}} \beta$ for some formulas α and β in the language of $\mathscr{L}$.*

Remark 1. *If $\mathscr{L}$ has a deductive implication $\rightarrow$, in the sense that it satisfies the Deduction meta-theorem DMT, then $\mathscr{L}$ is paraconsistent if and only if the schema formula $\varphi \rightarrow (\neg\varphi \rightarrow \psi)$ is not valid, i.e., the explosion law is not valid in $\mathscr{L}$ with respect to the negation $\neg$. That is, the negation $\neg$ is not explosive.*

Now, we present the notion of Logic of Formal Inconsistency.

Definition 5 (Logic of Formal Inconsistency)**.** *Let $\mathscr{L} = \langle \Theta, \vdash \rangle$ be a standard logic. Assume that the signature Θ of $\mathscr{L}$ contains a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Accordingly, $\mathscr{L}$ is a Logic of Formal Inconsistency, (**LFI**), with respect to $\neg$ and $\bigcirc(p)$ if the following holds:*

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ;

(ii) There are two formulas α and β such that:

  (a) $\bigcirc(\alpha), \alpha \nvdash \beta$;

  (b) $\bigcirc(\alpha), \neg\alpha \nvdash \beta$;

(iii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

Remark 2.

— *When $\bigcirc$ is a singleton, its elements are denoted by $\circ p$, where $\circ$ is the consistency operator.*

— *A logic that satisfies the property (iii) is called gently explosive.*

Finally, we define a stronger notion of **LFI**s for more reference, see [9].

Definition 6 (Strong Logic of Formal Inconsistency)**.** *Let $\mathscr{L} = \langle \Theta, \vdash \rangle$ be a standard logic. Assume that the signature Θ of $\mathscr{L}$ contains a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Then $\mathscr{L}$ is a strong **LFI** with respect to $\neg$ and $\bigcirc(p)$ if the following holds:*

(i) there are two formulas α and β such that:

  (a) $\alpha, \neg\alpha \nvdash \beta$;

  (b) $\bigcirc(\alpha), \alpha \nvdash \beta$;

  (c) $\bigcirc(\alpha), \neg\alpha \nvdash \beta$; and

(ii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

Remark 3.

— *Any strong **LFI** is an **LFI**.*

— *If $\mathscr{L}$ is a propositional logic then $\mathscr{L}$ is a strong **LFI** whenever the following holds:*

  (i) there are two formulas p and q such that:

    (a) $p, \neg p \nvdash q$;

    (b) $\bigcirc(p), p \nvdash q$;

    (c) $\bigcirc(p), \neg p \nvdash q$; and

  (ii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

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Definition 7 (Blok and Pigozzi algebraizability)**.** *Let Θ be a propositional signature, and let $\mathscr{L}$ be a standard propositional logic defined over the language $\mathcal{L}_\Theta$, with a consequence relation $\vdash_\mathcal{L}$. Then $\mathscr{L}$ is algebraizable in the sense of Blok and Pigozzi if there exists a nonempty set $\Delta(p_1, p_2) \subseteq \mathcal{L}_\Theta$ of formulas depending on variables p_1 and p_2, and a nonempty set $E(p_1) \subseteq \mathcal{L}_\Theta \times \mathcal{L}_\Theta$ of pairs of formulas depending on variable p_1 satisfying the following properties:*

(i) $\vdash_\mathscr{L} \delta(p_1, p_1)$, *for every* $\delta(p_1, p_2) \in \Delta(p_1, p_2)$;

(ii) $\Delta(p_1, p_2) \vdash_\mathscr{L} \delta(p_2, p_1)$, *for every* $\delta(p_1, p_2) \in \Delta(p_1, p_2)$;

(iii) $\Delta(p_1, p_2), \Delta(p_2, p_3) \vdash_\mathscr{L} \delta(p_1, p_3)$, *for every* $\delta(p_1, p_2) \in \Delta(p_1, p_2)$;

(iv) $\Delta(p_1, p_{n+1}), \ldots, \Delta(p_n, p_{2n}) \vdash_\mathscr{L} \delta(\#(p_1, \ldots, p_n), \#(p_{n+1}, \ldots, p_{2n}))$, *for every* $\delta(p_1, p_2) \in \Delta(p_1, p_2)$, *every n-ary connective* $\#$ *of* Θ *and every* $n \geq 1$;

(v) $p_1 \vdash_\mathscr{L} \delta(\gamma(p_1), \epsilon(p_1))$, *for every* $\delta(p_1, p_2) \in \Delta(p_1, p_2)$ *and every* $\langle \gamma(p_1), \epsilon(p_1) \rangle \in E(p_1)$;

(vi) $\{\delta(\gamma(p_1), \epsilon(p_1)) : \delta(p_1, p_2) \in \Delta(p_1, p_2), \langle \gamma(p_1), \epsilon(p_1) \rangle \in E(p_1)\} \vdash_\mathscr{L} p_1$.

The sets $\Delta(p_1, p_2)$ and $E(p_1)$ are called systems of equivalence formulas and defining equations, respectively.

Definition 8 (Relation)**.** *Let Θ be a propositional signature, and let $\theta \subseteq \mathcal{L}_\Theta \times \mathcal{L}_\Theta$ be a relation defined over the algebra of formulas $\mathcal{L}_\Theta$ if it satisfies the following properties:*

(i) $\alpha\theta\alpha$ *for every* $\alpha \in \mathcal{L}_\Theta$ *(reflexivity);*

(ii) $\alpha\theta\beta$ *implies* $\beta\theta\alpha$ *for every* $\alpha, \beta \in \mathcal{L}_\Theta$ *(symmetry);*

(iii) $\alpha\theta\beta$ *and* $\beta\theta\gamma$ *implies* $\alpha\theta\gamma$ *for every* $\alpha, \beta\gamma \in \mathcal{L}_\Theta$ *(transitivity);*

(iv) Given α_i *and* β_i *in* $\mathcal{L}_\Theta$ *(for* $1 \leq i \leq n$*) such that* $\alpha_1\theta\beta_1, \ldots, \alpha_n\theta\beta_n$, *then* $\#(\alpha_1, \ldots, \alpha_n)\theta\#(\beta_1, \ldots, \beta_n)$ *for every n-ary connective* $\#$ *of* Θ *and every* $n \geq 1$.

A congruence θ in $\mathcal{L}_\Theta$ is trivial if either $\theta = \mathcal{L}_\Theta \times \mathcal{L}_\Theta$ or $\theta = \{(\alpha, \alpha) : \alpha \in \mathcal{L}_\Theta\}$.

Definition 9 (Logical congruence)**.** *Let $\mathscr{L}$ be a standard logic defined over the language $\mathcal{L}_\Theta$.*

(i) A congruence θ in $\mathcal{L}_\Theta$ is compatible with a theory $\Gamma \subseteq \mathcal{L}_\Theta$ if it satisfies the following:

$$\alpha\theta\beta \text{ and } \Gamma \vdash_\mathscr{L} \alpha \text{ implies that } \Gamma \vdash_\mathscr{L} \beta.$$

(ii) A congruence θ in $\mathcal{L}_\Theta$ is a logical congruence in $\mathscr{L}$ if θ is compatible with every theory Γ. Equivalently, θ is a logical congruence in $\mathscr{L}$ if, for every α and β:

$$\alpha\theta\beta \text{ implies that } \alpha \vdash_\mathscr{L} \beta \text{ and } \beta \vdash_\mathscr{L} \alpha.$$

The usual mode to define the many-valued semantics of logic is through a matrix. We introduce the definition of the deterministic matrix, also known as the logical matrix or just as a matrix. In [19], we can find and an exhaustive discussion about many-valued logic and some examples.

Definition 10 (Matrix)**.** *Given a logic $\mathscr{L}$ in the language $\mathcal{L}$, the matrix of $\mathscr{L}$ is a structure $M = \langle D, D^*, F \rangle$, where:*

(i) D *is a non-empty set of truth values (domain),*

(ii) D^* *is a subset of* D *(set of designated values),*

(iii) $F = \{f_c | c \in \mathcal{C}\}$ *is a set of truth functions, with one function for each logical connective* c *of* $\mathcal{L}$.

Definition 11 (Interpretation)**.** *Given a logic $\mathscr{L}$ in the language $\mathcal{L}$, an interpretation t, is a function $t : Var \to D$ that maps propositional variables to elements in the domain.*

Any interpretation t can extend to a function on all formulas in $\mathcal{L}_\Sigma$ as usual, i.e., applying recursively the truth functions of logical connectives in F. If t is a valuation in the logic $\mathscr{L}$, we will say that t is an $\mathscr{L}$-valuation. Interpretations allow us to define the notion of validity in this type of semantics as follows:

Definition 12 (Valid formula)**.** *Given a formula φ and an interpretation t in a logic $\mathscr{L}$, we say that the formula φ is valid under t in $\mathscr{L}$, if $t(\varphi) \in D^*$, and we denote it as $t \models_\mathscr{L} \varphi$.*

Let us note that validity depends on the interpretation, but if we want to talk about "logical truths" in the system, then the validity should be absolute, as stated in the following definition:

Definition 13 (Tautology)**.** *Given a formula φ in the language of a logic $\mathscr{L}$, we say φ is a tautology in $\mathscr{L}$, if for every possible interpretation, the formula φ is valid, and we denote it as $\models_{\mathscr{L}} \varphi$.*

If φ is a tautology in the logic $\mathscr{L}$, we say that φ is an $\mathscr{L}$-tautology. When logic defined via a many-valued semantics, it is common to define the set of theorems of $\mathscr{L}$ as the set of tautologies obtained from the many-valued semantics, i.e., $\varphi \in \mathscr{L}$ if and only if $\models_{\mathscr{L}} \varphi$.

Definition 14 (Translation between Logics)**.** *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logics with sets of formulas $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. A mapping $* : \mathcal{L}_1 \to \mathcal{L}_2$ is said to be a translation from $\mathscr{L}_1$ to $\mathscr{L}_2$ if, for every $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_1$:*

$$\Gamma \models_{\mathscr{L}_1} \alpha \text{ then } \Gamma^* \models_{\mathscr{L}_1} \alpha^*.$$

And it is said to be a conservative translation if if satisfies the stronger property:

$$\Gamma \models_{\mathscr{L}_1} \alpha \text{ if and only if } \Gamma^* \models_{\mathscr{L}_1} \alpha^*.$$

If $*$ is a mapping defined on formulas and Γ is a set of formulas, then $\Gamma^* = \{\gamma^* : \gamma \in \Gamma\}$.

3 Blok-Pigozzi Algebrization

In this section, we study the $\mathbf{CG}'_3$; and $\mathbf{G}'_3$ logics, which are defined in terms of four connectives $\wedge$, $\vee$, $\to$, and $\neg$ where the implication is deductive to $\mathbf{CG}'_3$. We establish that $\mathbf{CG}'_3$ and $\mathbf{G}'_3$ are algebraizable logics with the Blok-Pigozzi method.

Table 1. Truth functions of the connectives in $\mathbf{CG}'_3$ and $\mathbf{G}'_3$

$\vee$	**0**	**1**	**2**
0	0	1	2
1	1	1	2
2	2	2	2

$\wedge$	**0**	**1**	**2**
0	0	0	0
1	0	1	1
2	0	1	2

$\to$	**0**	**1**	**2**
0	2	2	2
1	0	2	2
2	0	1	2

	$\neg$
0	2
1	2
2	0

3.1 The $\mathbf{CG}'_3$ logic

This section aims to analyze the algebraizability of $\mathbf{CG}'_3$ in the sense of Blok and Pigozzi, and we see the $\mathbf{CG}'_3$ logic as a Logic of Formal Inconsistency (**LFI**).

Osorio et al. defined $\mathbf{CG}'_3$ logic as a three-valued logic in [21], where the matrix is giving by the structure $\mathcal{M} = \langle D, D^*, F\rangle$ over $\Sigma = \{\vee, \wedge, \to, \neg\}$, where $D = \{0, 1, 2\}$, the set $\mathcal{D}^*$ of designated values is $\{1, 2\}$, and $\mathcal{F}$ is the set of truth functions defined in Table 1.

Remark 4.

— *Observe that $\to$ is a deductive implication: $\Gamma, \alpha \models_{\mathbf{CG}'_3} \beta$ if and only if $\Gamma \models_{\mathbf{CG}'_3} \alpha \to \beta$.*

— *Considering the natural order $0 \leq 1 \leq 2$ in D, the $\vee$ corresponds to the supremum, $\wedge$ corresponds to the infimum and $\to$ is the residuum of $\wedge$:*

$$z \wedge x \leq y \text{ if and only if } z \leq x \to y,$$

$$\text{for every } x, y, z \in D.$$

— *The $\mathbf{CG}'_3$ logic was axiomatized in [23] applying the Lindenbaum-Łos method. Furthermore, the authors define two connectives (strong negation and inconsistency operator):*

1. *$\sim\varphi = \varphi \to \bot$ (Strong negation),*
2. *$\bullet\varphi = \sim\sim\varphi \wedge \neg\varphi$ (inconsistency operator).*

Table 2. Truth functions of the connectives $\sim, \bullet$, and $\circ$ in $\mathbf{CG}'_3$

	$\sim$
0	2
1	0
2	0

	$\bullet$
0	0
1	2
2	0

	$\circ$
0	2
1	0
2	2

We define the inconsistency operator as follows:

3. $\circ\varphi = \neg \bullet \varphi$ *(consistency operator)*

Truth functions for the connectives $\sim, \bullet, \circ$ in $\mathbf{CG}'_3$ are displayed in Table 2.

In [6], Blok and Pigozzi gave a mathematical concept of algebraizable logic. The main idea of this definition is the following:

A logic is algebraizable if there exists a class of algebras associated with the system of reasoning. In the same way as the class of Boolean algebras is related to classical propositional logic.

Proposition 1. *The logic $\mathbf{CG}'_3$ is a strong **LFI** with consistency operator $\circ$ defined as above.*

Proof. Assume that p and q are two different propositional variables. By considering the valuation v_1 such that $v_1(p) = 1$, $v_1(\neg p) = 2$, and $v_1(q) = 0$, it follows that p, $\neg p \nvdash_{\mathbf{CG}'_3} q$ and clause $(i.a)$ of Remark 3 is verified. Consider the valuation v_2 such that $v_2(p) = 2$, $v_2(\circ p) = 2$, and $v_2(q) = 0$, it follows that p, $\circ p \nvdash_{\mathbf{CG}'_3} q$ and clause $(i.b)$ of Remark 3 is satisfied. Now, considering the valuation v_3 such that $v_3(\neg p) = 1$, $v_3(\circ p) = 2$, and $v_3(q) = 0$, it follows that $\neg p$, $\circ p \nvdash_{\mathbf{CG}'_3} q$ and clause $(i.c)$ of Remark 3 is verified. Finally, there is no valuation that makes formulas φ, $\neg\varphi$, and $\circ\varphi$ simultaneously true. Thus, item (ii) of Remark 3 is satisfied. Thus $\mathbf{CG}'_3$ is a strong **LFI** w.r.t $\neg$ and $\circ$. □

Proposition 2. *Let h be a valuation for $\mathbf{CG}'_3$. Then:*

(i) $h(p_1 \to p_2) \in D^*$ *if and only if* $h(p_1) = 0$ *or* $h(p_2) \in D^*$*;*

(ii) $h(p_1 \wedge p_2) \in D^*$ *if and only if* $h(p_1) \in D^*$ *and* $h(p_2) \in D^*$*;*

(iii) $h(p_1 \vee p_2) \in D^*$ *if and only if* $h(p_1) \in D^*$ *or* $h(p_2) \in D^*$*;*

(iv) $h(p_1 \leftrightarrow p_2) \in D^*$ *if and only if either* $h(p_1) \in D^*$ *and* $h(p_2) \in D^*$*, or* $h(p_1) = h(p_2) = 0$*.*

Proof. Immediate from the truth-tables. □

Definition 15. *Let $\delta(p_1, p_2)$ be the following formula of $\mathcal{L}_\Sigma$:*

$$\delta(p_1, p_2) = (p_1 \leftrightarrow p_2) \wedge (\circ p_1 \leftrightarrow \circ p_2).$$

Proposition 3. *Let α and β be formulas in $\mathcal{L}_\Sigma$. Then for every valuation h for $\mathbf{CG}'_3$ it holds that:*

1. $h(\delta(\alpha, \beta)) \in D^*$ *if and only if we have:*

   (a) *either* $h(\alpha) \in D^*$ *and* $h(\beta) \in D^*$*, or* $h(\alpha) = h(\beta) = 0$*; and*

   (b) $h(\circ\alpha) = h(\circ\beta)$*.*

   Thus, $h(\delta(\alpha, \beta)) \in D^*$ *if and only if* $h(\alpha) = h(\beta)$*.*

2. $h(\bullet\alpha \to \alpha) \in D^*$*.*

3. $h(\circ\alpha) = h(\circ(\bullet\alpha \to \alpha))$*.*

4. $h(\delta(\alpha, \bullet\alpha \to \alpha)) \in D^*$ *if and only if* $h(\alpha) \in D^*$*.*

Proof.

1. "**Only if**" part. $h(\delta(\alpha, \beta)) \in D^*$ if and only if in accordance with Proposition 2(ii), $h(\alpha \leftrightarrow \beta) \in D^*$ and $h(\circ\alpha \leftrightarrow \circ\beta) \in D^*$. By Proposition 2(iv), $h(\alpha \leftrightarrow \beta) \in D^*$ is equivalent to either $h(\alpha) \in D^*$ and $h(\beta) \in D^*$, or $h(\alpha) = h(\beta) = 0$, while $h(\circ\alpha \leftrightarrow \circ\beta) \in D^*$ if and only if either $h(\circ\alpha) \in D^*$ and $h(\circ\beta) \in D^*$, or $h(\circ\alpha) = h(\circ\beta) = 0$. By the definition of $\circ$, $h(\circ\alpha \leftrightarrow \circ\beta) \in D^*$ is equivalent to $h(\circ\alpha) = h(\circ\beta)$. Now, suppose that $h(\delta(\alpha, \beta)) \in D^*$. If $h(\alpha) = 2$ and $h(\beta) = 1$, then $h(\circ\alpha) = 2$ and $h(\circ\beta) = 0$, violating that $h(\circ\alpha) = h(\circ\beta)$. Analogously, it is impossible to have $h(\alpha) = 1$ and $h(\beta) = 2$. This shows that $h(\alpha) = h(\beta)$. "**If**" part is obvious, in light of the clauses.

2. The proof is straightforward.

3. If $h(\circ\alpha) = 2$ then $h(\alpha) \in \{0, 2\}$ and $h(\bullet\alpha) = 0$ and so $h(\bullet\alpha \to \alpha) = 2$; hence, $h(\circ(\bullet\alpha \to \alpha)) = 2$. If $h(\circ\alpha) = 0$ then $h(\alpha) = 1$ and so $h(\bullet\alpha) = 2$; hence, $h(\bullet\alpha \to \alpha) = 1$; therefore, $h(\circ(\bullet\alpha \to \alpha)) = 0$.

4. "**Only if**" part. By item 1, $h(\delta(\alpha, \bullet\alpha \to \alpha)) \in D^*$ implies that $h(\alpha) = h(\bullet\alpha \to \alpha)$. So, by item 2, $h(\alpha) \in D^*$. "**If**" part. Suppose that $h(\alpha) \in D^*$. By item 2, $h(\bullet\alpha \to \alpha) \in D^*$ and, by item 3, $h(\circ\alpha) = h(\circ(\bullet\alpha \to \alpha))$. Finally, applying item 1, $h(\delta(\alpha, \bullet\alpha \to \alpha)) \in D^*$.

□

Theorem 1. *The logic $\mathbf{CG}_3'$ is algebraizable in the sense of Blok and Pigozzi with a system of equivalence formulas given by $\Delta(p_1, p_2) = \{\delta(p_1, p_2)\}$ and a system of defining equations given by $E(p_1) = \{p_1, \bullet p_1 \to p_1\}$.*

Proof. It is easy to prove that the system $\Delta(p_1, p_2)$ satisfies conditions (i)-(iv) by item 1 of Proposition 3. By item 4 of the same proposition, conditions (v)-(vi) follow easily. □

3.2 The $\mathbf{G}_3'$ Logic

The logic $\mathbf{G}_3'$ is defining as a three-valued logic in [22]. The matrix is giving by the structure $\mathcal{M} = \langle D, D^*, F\rangle$ over $\Sigma = \{\vee, \wedge, \to, \neg\}$, where $D = \{0, 1, 2\}$, the set $\mathcal{D}^*$ of designated values is $\{2\}$, and $\mathcal{F}$ is the set of truth functions defined in Table 1. Note that $\mathbf{G}_3'$ is defined in the signature of $\mathbf{CG}_3'$.

It is easy to check from truth functions that $\models_{\mathbf{CG}_3'} ((\alpha \to \beta) \to \alpha) \to \alpha$ but $\not\models_{\mathbf{G}_3'} ((\alpha \to \beta) \to \alpha) \to \alpha$. So the two deductive systems are different.

We now define a mapping $\mathbf{CG}_3'$ into $\mathbf{G}_3'$.

Definition 16. *Let the mapping $* : \mathcal{L}_\Sigma \to \mathcal{L}_\Sigma$ given by $\alpha^* = {\sim}\alpha \to \alpha$.*

We now show the mapping defined above is a conservative translation.

Proposition 4. *For every $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_\Sigma$:*

$$\Gamma \models_{\mathbf{CG}_3'} \alpha \text{ if and only if } \Gamma^* \models_{\mathbf{G}_3'} \alpha^*.$$

Proof. "**Only if**" part. Suppose otherwise, that is, $\Gamma \models_{\mathbf{CG}_3'} \alpha$ and $\Gamma^* \not\models_{\mathbf{G}_3'} \alpha^*$. Then there are a valuation h such that $h(\Gamma^*) \in D^*$ and $h(\alpha^*) = 0$, then $h({\sim}\alpha) \in \{1, 2\}$ and $h(\alpha) = 0$, but $\Gamma \models_{\mathbf{CG}_3'} \alpha$ then for that valuation $h(\alpha) \in D^*$, is a contradiction. "**If**" part. $\Gamma^* \models_{\mathbf{G}_3'} \alpha^*$ implies $\Gamma \models_{\mathbf{CG}_3'} \alpha^*$, hence, using the following valid formula $({\sim}\alpha \to \alpha) \to \alpha$ in $\mathbf{CG}_3'$, we obtain $\Gamma \models_{\mathbf{CG}_3'} \alpha$. □

Corollary 1. *$\mathbf{G}_3'$ is algebraizable in the sense of Blok and Pigozzi.*

The systems $\mathbf{CG}_3'$ and $\mathbf{G}_3'$ despite, being different, are algebraizable in the sense of Blok and Pigozzi.

4 Conclusions and Future Work

$\mathbf{CG}_3'$ is defined by multi-valued semantics. The logic matrix is given by $\mathbf{CG}_3'$ is given by $M = (D, D*, F)$; where the domain is $D = \{0, 1, 2\}$ and the set of values designated is $D^* = \{1, 2\}$. This logic is paraconsistent and can be seen as an extension of the $\mathbf{G}_3'$ logic, introduced by Osorio in 2008 [22]. In this article, we expanded the studies on these logics, in particular, we showed some results related to algebraic logic. The main result of the work is the algebrization using the Blok and Pigozzi technique.

Algebraic semantics of logical languages implies generality and compositionality in the design, implementation and maintenance of reasoning systems [10].

Among the applications, it is of our particular attention the verification of systems [18]. Another research question of our interest is regarding the relationship of $\mathbf{G}_3'$ and $\mathbf{CG}_3'$ and the annotated paraconsistent logics family, defined by Subrahmanian in [24].

Nowadays, many applications to paraconsistent logic are known in many fields of computer science, such as electrical circuits, non-monotonous reasoning, control systems, automation, and robotics, to mention a few [4].

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